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MATHEMATICS
TARGET IIT JEE

QUADRATIC EQUATION

THEORY AND EXERCISE BOOKLET

CONTENTS

JEE Syllabus :

Quadratic equations with real coefficients, relations between roots and coefficients, formation of quadratic equations with given roots, symmetric functions of roots

A. GENERAL POLYNOMIAL

A function f defined by $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots\ldots\ldots + a_1 x + a_0$, where $a_0, a_1, a_2, \ldots\ldots\ldots, a_n \in R$ is called **n degree polynomial** while coefficient ($a_n \neq 0$, $n \in W$) is real. If $a_0, a_1, a_2, \ldots\ldots\ldots, a_n \in C$, then it is called **complex cofficient polynomial**.

B. QUADRATIC POLYNOMIAL

A polynomial of degree two in one variable $f(x) = y = ax^2 + bx + c$, where $a \neq 0$ & $a, b, c \in R$
$a \rightarrow$ leading coefficient, $c \rightarrow$ absolute term / constant term
If $a = 0$ then $y = bx + c \rightarrow$ linear polynomial $b \neq 0$
If $a = 0$, $c = 0$ then $y = bx \rightarrow$ odd linear polynomial

C. QUADRATIC EQUATION

1. The solution of the quadratic equation , $ax^2 + bx + c = 0$ is given by $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

The expression $b^2 - 4ac = D$ is called the discriminant of the quadratic equation.

2. If α & β are the roots of the quadratic equation $ax^2 + bx + c = 0$, then ;

(i) $\alpha + \beta = -b/a$ **(ii)** $\alpha\beta = c/a$ **(iii)** $\alpha - \beta = \sqrt{D}/a$.

D. NATURE OF ROOTS

(1) Consider the quadratic equation $ax^2 + bx + c = 0$ where $a, b, c \in R$ & $a \neq 0$ then ;

(i) $D > 0 \Leftrightarrow$ roots are real & distinct (unequal)
(ii) $D = 0 \Leftrightarrow$ roots are real & coincident (equal)
(iii) $D < 0 \Leftrightarrow$ roots are imaginary
(iv) If $p + iq$ is one root of a quadratic equation, then the other must be the conjugate $p - iq$ & vice versa. ($p, q \in R$ & $i = \sqrt{-1}$)

(2) Consider the quadratic equation $ax^2 + bx + c = 0$ where $a, b, c \in Q$ & $a \neq 0$ then ;

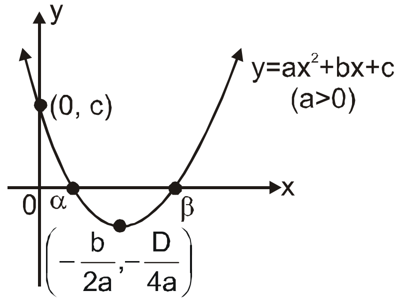
(i) If $D > 0$ & is a perfect square , then roots are rational & unequal .
(ii) If $\alpha = p + \sqrt{q}$ is one root in this case, (where p is rational & $\sqrt{q}$ is a surd) then the other root must be the conjugate of it i.e. $\beta = p - \sqrt{q}$ & vice versa.

Consider the quadratic equation, $ax^2 + bx + c = 0$ having α, β as its roots ; $D \equiv b^2 - 4ac$

- $D = 0$
  Roots are equal $\alpha = \beta = -b/2a$
- $D \neq 0$
  Roots are unequal
  - $a, b, c \in R$ & $D > 0$
    Roots are real
    - $a, b, c \in Q$ & D is perfect square
      $\Rightarrow$ Roots are rational
      - $a = 1$, $b, c \in I$ & D is perfect square
        $\Rightarrow$ Roots are integral.
    - $a, b, c \in Q$ & D is not a perfect square
      $\Rightarrow$ Roots are irrational i.e. $\alpha = p + \sqrt{q}, \beta = p - \sqrt{q}$
  - $a, b, c \in R$ & $D < 0$
    Roots are imaginary $\alpha = p + iq$, $\beta = p - iq$

E. GRAPH OF QUADRATIC EXPRESSION

$$y = f(x) = a\,x^2 + b\,x + c \text{ or } \left(y + \frac{D}{4a}\right) = a\left(x + \frac{b}{2a}\right)^2$$



(i) The graph between x , y is always a parabola.

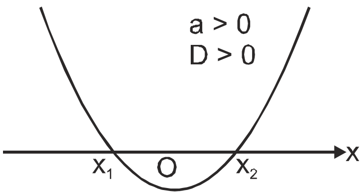
(ii) If a > 0 then the shape of the parabola is concave upwards & if a < 0 then the shape of the parabola is concave downwards .

(iii) The co–ordinate of vertex are $\left(-\frac{b}{2a}, -\frac{D}{4a}\right)$

(iv) The parabola intersect the y–axis at point (0, c)

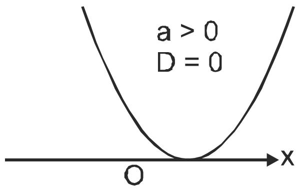
(v) The x–co–ordinate of point of intersection of parabola with x–axis are the real roots of the quadratic equation f(x) = 0. Hence the parabola may or may not intersect the x–axis at real points.

Fig. 1



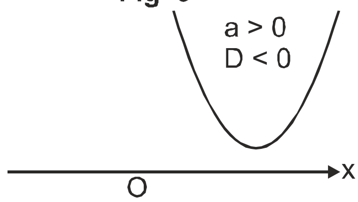
Roots are real & distinct

Fig. 2



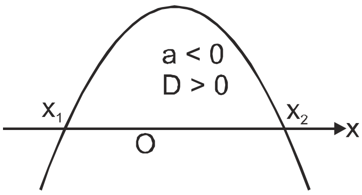
Roots are coincident

Fig 3



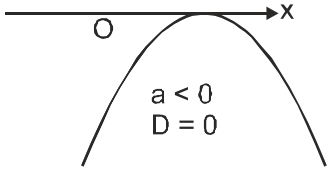
Roots are complex conjugate

Fig. 4



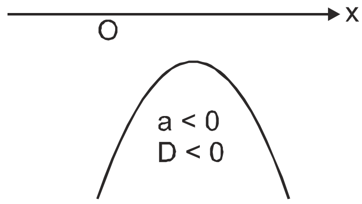
Roots are real & distinct

Fig. 5



Roots are coincident

Fig. 6



Roots are complex conjugate

Consider the quadratic expression , $y = ax^2 + bx + c$, $a \neq 0$ & a , b , c $\in$ R then ;

(i) $\forall$ $x \in R$, y > 0 only if a > 0 & $b^2 - 4ac < 0$ (figure 3).

(ii) $\forall$ $x \in R$, y < 0 only if a < 0 & $b^2 - 4ac < 0$ (figure 6).

F. RELATION BETWEEN ROOTS & COEFFICIENTS

A quadratic equation whose roots are α & β is $(x - \alpha)(x - \beta) = 0$

i.e. $x^2 - (\alpha + \beta)\,x + \alpha\beta = 0$ i.e. x^2 – (sum of roots) x + product of roots = 0

Ex.1 A quadratic polynomial p(x) has $1 + \sqrt{5}$ and $1 - \sqrt{5}$ as roots and it satisfies p(1) = 2. Find the quadratic polynomial.

Sol. sum of the roots = 2, product of the roots = – 4

$\therefore$ let $p(x) = a(x^2 - 2x - 4) \Rightarrow p(1) = 2 \Rightarrow 2 = a(1^2 - 2 \cdot 1 - 4) \Rightarrow a = -2/5$

$\therefore$ $p(x) = -2/5\,(x^2 - 2x - 4)$

Ex.2 The quadratic equation $x^2 + mx + n = 0$ has roots which are twice those of $x^2 + px + m = 0$ and m, n and $p \neq 0$. Find the value of $\frac{n}{p}$.

Sol. $x^2 + mx + n = 0 \begin{cases} 2\alpha \\ 2\beta \end{cases}$ and $x^2 + px + n = 0 \begin{cases} \alpha \\ \beta \end{cases}$

$2(\alpha + \beta) = -m$ ....(1) $\quad 4\alpha\beta = n$ ....(2)

and $\alpha + \beta = -p$ ....(3) $\quad \alpha\beta = m$ ....(4)

$\therefore$ (1) and (3) $\Rightarrow$ $2p = m$ and (2) and (4) $\Rightarrow$ $4m = n$

$\Rightarrow \frac{n}{p} = \frac{4m}{m/2} = 8$

Ex.3 Find the range of the variable x satisfying the quadratic equation,

$x^2 + (2\cos\phi)x - \sin^2\phi = 0 \quad \forall \ \phi \in R$.

Sol. Roots of the equation $x^2 + (2\cos\phi)x - \sin^2\phi = 0$ are $x_1 = -\cos\phi + 1$ or $x_2 = -(1 + \cos\phi)$

hence $x \in [0, 2]$ or $x \in [-2, 0]$ $\quad \therefore x \in [-2, 2]$

Ex.4 If α & β are the roots of the equation $x^2 - ax + b = 0$ and $v_n = \alpha^n + \beta^n$, show that $v_{n+1} = a v_n - b v_{n-1}$ and hence obtain the value of $\alpha^5 + \beta^5$.

Sol. $\alpha + \beta = a$; $\alpha\beta = b$; $v_n = \alpha^n + \beta^n$

$v_{n+1} = \alpha^{n+1} + \beta^{n+1} = (\alpha + \beta)(\alpha^n + \beta^n) - \alpha\beta^n - \beta\alpha^n = a v_n - \alpha\beta(\alpha^{n-1} + \beta^{n-1})$

Now $\alpha^5 + \beta^5 = v_5 = a v_4 - b v_3 = a(a v_3 - b v_2) - b v_3 = (a^2 - b) v_3 - ab v_2$

$= (a^2 - b)[a v_2 - b v_1] - ab v_2 = (a(a^2 - b) - ab) v_2 - b(a^2 - b) v_1$

$= (a^3 - 2ab)(a^2 - 2b) - ab(a^2 - b) = a^5 - 2a^3b - 2a^3b + 4ab^2 - a^3b + ab^2$

$= a^5 - 5a^3b + 5ab^2 \quad \Rightarrow v_{n+1} = a v_n - b v_{n-1}$

Ex.5 Let p & q be the two roots of the equation, $mx^2 + x(2 - m) + 3 = 0$. Let m_1, m_2 be the two values of m satisfying $\frac{p}{q} + \frac{q}{p} = \frac{2}{3}$. Determine the numerical value of $\frac{m_1}{m_2^2} + \frac{m_2}{m_1^2}$.

Sol. $mx^2 + (2 - m)x + 3 = 0 \begin{cases} p \\ q \end{cases} \Rightarrow p + q = \frac{m-2}{m}$; $pq = \frac{3}{m}$

now m_1 and m_2 satisfies $\frac{p}{q} + \frac{q}{p} = \frac{2}{3} \Rightarrow \frac{p^2+q^2}{pq} = \frac{2}{3}$

$\frac{(p+q)^2 - 2pq}{pq} = \frac{2}{3} \Rightarrow \left(\frac{m-2}{m}\right)^2 - \frac{6}{m} = \frac{2}{3}.\frac{3}{m} = \frac{2}{m} \Rightarrow \frac{(m-2)^2}{m^2} = \frac{8}{m}$

$m^2 - 4m + 4 = 8m \Rightarrow m^2 - 12m + 4 = 0 \quad \therefore \quad m_1 + m_2 = 12$ and $m_1 m_2 = 4$

now $\frac{m_1}{m_2^2} + \frac{m_2}{m_1^2} = \frac{m_1^3 + m_2^3}{(m_1 m_2)^2} = \frac{(m_1+m_2)^3 - 3m_1m_2(m_1+m_2)}{(m_1m_2)^2} = \frac{12^3 - 12.12}{16} = \frac{12^2.11}{16} = 99$

Ex.6 One root of $mx^2 - 10x + 3 = 0$ is two third of the other root. Find the sum of the roots.

Sol. $\alpha + \frac{2\alpha}{3} = \frac{10}{m} \Rightarrow \frac{5\alpha}{3} = \frac{10}{m} \Rightarrow \alpha = \frac{6}{m}$;

also $\alpha \cdot \frac{2\alpha}{3} = \frac{3}{m} \Rightarrow 2\alpha^2 = \frac{9}{m} \Rightarrow 2\cdot\frac{36}{m^2} = \frac{9}{m} \Rightarrow \frac{24}{m} = 3 \Rightarrow m = 8$ $\quad\therefore$ sum $= \frac{10}{m} = \frac{10}{8} = \frac{5}{4}$

Ex.7 If $x_1 \in N$ and x_1 satisfies the equation. If $x^2 + ax + b + 1 = 0$, where a, $b \neq -1$ are integers has a root in natural numbers then prove that $a^2 + b^2$ is a composite.

Sol. Let α and β be the two roots of the equation where $\alpha \in N$. Then

$\alpha + \beta = -a$ ...(1) $\quad \alpha . \beta = b + 1$ ...(2)

$\therefore$ $\beta = -a - \alpha$ is an integer. Also, since $b + 1 \neq 0$, $\beta \neq 0$.

From eq. (1) & eq. (2), we get $a^2 + b^2 = (\alpha+\beta)^2 + (\alpha\beta - 1)^2 = \alpha^2 + \beta^2 + \alpha^2\beta^2 + 1 = (1 + \alpha^2)(1 + \beta^2)$

Now, as $\alpha \in N$ and β is a non–zero integer, $1 + \alpha^2 > 1$ and $1 + \beta^2 > 1$.

Hence $a^2 + b^2$ is composite number

Note : If $b = -1$, then $a^2 + b^2$ can not be a composite number.

Consider $a = -6$, $\quad b = -1$

$x^2 - 6x + (-1) + 1 = 0$, its are 6 and 0.

$a^2 + b^2 = 36 + 1 = 37$, a prime number.

Ex.8 Find a quadratic equation whose roots x_1 and x_2 satisfy the condition

$x_1^2 + x_2^2 = 5, \quad 3(x_1^5 + x_2^5) = 11(x_1^3 + x_2^3)$. (Assume that x_1, x_2 are real)

Sol. We have $3(x_1^5 + x_2^5) = 11(x_1^3 + x_2^3) \Rightarrow \frac{x_1^5 + x_2^5}{x_1^3 + x_2^3} = \frac{11}{3} \Rightarrow \frac{(x_1^2 + x_2^2)(x_1^3 + x_2^3) - x_1^2x_2^2(x_1 + x_2)}{(x_1^3 + x_2^3)} = \frac{11}{3}$

$\Rightarrow (x_1^2 + x_2^2) - \frac{x_1^2x_2^2(x_1 + x_2)}{(x_1 + x_2)(x_1^2 + x_2^2 - x_1x_2)} = \frac{11}{3} \Rightarrow 5 - \frac{x_1^2x_2^2}{5 - x_1x_2} = \frac{11}{3} \Rightarrow 3x_1^2x_2^2 + 4x_1x_2 - 20 = 0$

$\Rightarrow 3x_1^2x_2^2 + 10x_1x_2 - 6x_1x_2 - 20 = 0 \Rightarrow (x_1x_2 - 2)(3x_1x_2 + 10) = 0 \quad \therefore x_1x_2 = 2, -\frac{10}{3}$

We have $(x_1 + x_2)^2 = 5 + 4 = 9 \Rightarrow x_1 + x_2 = \pm 3$ (if $x_1x_2 = 2$)

$\therefore$ $(x_1 + x_2)^2 = 5 + 2(-10/3) = -5/3$ (if $x_1x_2 = -10/3$)

Which is not possible x_1, x_2 are real. Thus required quadratic equations are $x^2 \pm 3x + 2 = 0$

Ex.9 Form a quadratic equation with rational coefficients if one of its root is $\cot^2 18°$.

Sol. $\cot^2 18° = \frac{1+\cos 36°}{1-\cos 36°} = \frac{1+\frac{\sqrt{5}+1}{4}}{1-\frac{\sqrt{5}+1}{4}} = \frac{(5+\sqrt{5})(3+\sqrt{5})}{(3-\sqrt{5})(3+\sqrt{5})} = \frac{20+5\sqrt{5}+3\sqrt{5}}{9-5} = \frac{20+8\sqrt{5}}{4} = 5+2\sqrt{5}$

Hence if $\alpha = 5+2\sqrt{5}$, $\beta = 5-2\sqrt{5}$ $\quad \therefore \alpha + \beta = 10; \quad \alpha\beta = 25 - 20 = 5$

$\therefore$ quadratic equation is $x^2 - 10x + 5 = 0$

Ex.10 Let a & c be prime numbers and b an integer. Given that the quadratic equation $ax^2 + bx + c = 0$ has rational roots, show that one of the root is independent of the co–efficients. Find the two roots.

Sol. $b^2 - 4ac = n^2 \Rightarrow (b - n)(b + n) = 4ac$

$\left.\begin{array}{l}\text{Case I } \; b - n = 4a \text{ and } b + n = c \\ \text{Case II } \; b - n = 4c \text{ and } b + n = a\end{array}\right]$ not possible as $2b = 4a + c = \text{odd} \Rightarrow b$ is not an integer

Case III $b - n = 2a$ and $b + n = 2c \Rightarrow b = a + c$

Now $\alpha\beta = c/a$ & $\alpha + \beta = -b/a = -\dfrac{a+c}{a} = -1 - c/a \Rightarrow \alpha = -1$ and $\beta = -c/a$

Ex.11 If roots of the quadratic equation $x^2 - (2n + 18)x - n - 11 = 0$, $n \in$ set of integers, are rational, then find the value (s) of n.

Sol. Here discriminant of given equation must be perfect square.

i.e. $4[(n + 9)^2 + n + 11]$ must be perfect square $\Rightarrow n^2 + 19n + 92$ must be perfect square

$\Rightarrow n^2 + 19n + 92 = m^2 \Rightarrow n = \dfrac{-19 \pm \sqrt{4m^2 - 7}}{2} \Rightarrow 4m^2 - 7$ is a perfect square

$\Rightarrow 4m^2 - 7 = p^2 \Rightarrow 4m^2 - p^2 = 7 \Rightarrow (2m + p)(2m - p) = 7$

$\Rightarrow$ either $2m + p = \pm 1$, $2m - p = \pm 7$ or $2m + p = \pm 7$, $2m - p = \pm 1$

$\Rightarrow 2m = \pm 4 \Rightarrow m = \pm 2 \Rightarrow n^2 + 19n + 92 = 4 \Rightarrow (n + 8)(n + 11) = 0 \Rightarrow n = -8$ or -11

Ex.12 Find all integers values of a such that the quadratic expressions $(x + a)(x + 1991) + 1$ can be factored as $(x + b)(x + c)$, where b and c are integers.

Sol. $(x + a)(x + 1991) + 1 = (x + b)(x + c) \Rightarrow 1991 + a = b + c$ and $1991a + 1 = bc$

$\therefore (b - c)^2 = (b + c)^2 - 4bc = (1991 + a)^2 - 4(1991a + 1)$

$= \underbrace{(1991 + a)^2 - 4 \times 1991a} - 4 = (1991 - a)^2 - 4$ or $(1991 - a)^2 - (b - c)^2 = 4$

If the difference between two perfect square is 4, then one of them is 4 and the other is zero.

Therefore, $1991 - a = \pm 2$, $(b - c)^2 = 0$

$\Rightarrow a = 1991 + 2 = 1993$ and $b = c$ or $a = 1991 - 2 = 1989$ and $b = c$

But $b + c = 2b = 1991 + a = 1991 + 1993$ or $1991 + 1989 \Rightarrow b = c = 1992$ or 1990

So, the only 2 values of a are 1993 and 1989.

Ex.13 Find a, if $ax^2 - 4x + 9 = 0$ has integral roots.

Sol. Let $a = \dfrac{1}{b}$, so that the given equation becomes $x^2 - 4bx + 9b = 0$.

This equation has integeral roots if b is an integer and $16b^2 - 36b$ is a perfect square

Let $b(4b - 9) = k^2 \Rightarrow 4b^2 - 9b - k^2 = 0 \Rightarrow \left(2b - \dfrac{9}{4}\right)^2 - k^2 = \dfrac{81}{16} \Rightarrow (8b - 9)^2 - 16k^2 = 81$

$\Rightarrow (8b - 9 - 4k)(8b - 9 + 4k) = 81 = 3 \times 27$.

Since b and k are integers, $8b - 9 - 4k = 3$ and $8b - 9 + 4k = 27 \Rightarrow 16b - 18 = 30 \Rightarrow b = 3 \Rightarrow a = \dfrac{1}{3}$.

For any other factorization of 81, b will not be an integer.

Ex.14 If α, β are the roots of $x^2 + px + q = 0$ and also of $x^{2n} + p^n x^n + q^n = 0$, and if $\dfrac{\alpha}{\beta}, \dfrac{\beta}{\alpha}$ are the roots of $x^n + 1 + (x + 1)^n = 0$, then prove that n must be an even integer.

Sol. Since α, β are the roots of $x^2 + px + q = 0$ $\quad\therefore\quad$ $\alpha+\beta = -p$, $\alpha\beta = q$ ....(1)
Also α, β are roots of $x^{2n} + p^n x^n + q^n = 0$ $\quad\therefore\quad$ $\alpha^n + \beta^n = -p^n$ and $\alpha^n \beta^n = q^n$ ....(2)
Now, α/β are roots of $x^n + 1 + (x + 1)^n = 0$

$$\Rightarrow \frac{\alpha^n}{\beta^n} + 1 + \left(\frac{\alpha}{\beta} - 1\right)^n = 0 \qquad \Rightarrow \frac{\alpha^n + \beta^n}{\beta^n} + \frac{(\alpha+\beta)^n}{\beta^n} = 0$$

$\Rightarrow (\alpha^n + \beta^n) + (\alpha + \beta)^n = 0 \quad \Rightarrow \quad -p^n + (-p)^n = 0$ {from (1) and (2)}
This is true only if n is an even integer.

Ex.15 If p, q are the roots of the quadratic equation $x^2 + 2bx + c = 0$, prove that

$$2 \log\left(\sqrt{y-p} + \sqrt{y-q}\right) = \log 2 + \log\left(y + b + \sqrt{y^2 + 2by + c}\right)$$

Sol. $x^2 + 2bx + c = 0 \;<\; \begin{matrix} p \\ q \end{matrix}$; $\left.\begin{matrix} p+q=-2b \\ pq=c \end{matrix}\right\}$ ....(1)

TPT $2 \log\left(\sqrt{y-p} + \sqrt{y-q}\right) = \log 2 + \log\left(y + b + \sqrt{y^2 + 2by + c}\right)$

LHS = $\log\left(y - p + y - q + 2\sqrt{y-p}\sqrt{y-q}\right) = \log\left(2y - (p+q) + 2\sqrt{y^2 - (p+q)y + pq}\right)$

$= \log\left(2y + 2b + 2\sqrt{y^2 + 2by + c}\right) = \log 2 + \log\left(y + b + \sqrt{y^2 + 2by + c}\right) =$ RHS

Ex.16 Find all values of the parameter a for which the quadratic equation $(a+1)\, x^2 + 2(a + 1)\, x + a - 2 = 0$
(a) has two distinct roots, **(b)** has no roots, **(c)** has two equal roots.

Sol. By the hypothesis this equation is quadratic, and therefore $a \neq -1$ and the discriminant of this equation $D = 4\,(a+1)^2 - 4(a + 1)\,(a - 2) = 4\,(a + 1)\,(a + 1 - a + 2) = 12(a + 1)$. For $a > -1$, then $D > 0$ then this equation has two distinct roots. For $a < -1$ then $D < 0$, then this equation has no roots. This equation can not have two equal roots since $D = 0$ only for $a = -1$, and this contradicts the hypothesis.

Ex.17 If the equation $ax^2 + 2bx + c = 0$ has real roots, a, b, c being real numbers and if m and n are real numbers such that $m^2 > n > 0$ then prove that the equation $ax^2 + 2mbx + nc = 0$ has real roots.

Sol. Since roots of the equation $ax^2 + 2bx + c = 0$ are real $\therefore$ $(2b)^2 - 4ac \geq 0$ $\Rightarrow$ $b^2 - ac \geq 0$ ...(1)
and discriminant of $ax^2 + 2mbx + nc = 0$ is $D = (2mb)^2 - 4anc = 4m^2b^2 - 4anc$ ...(2)
from (1) $\quad b^2 \geq ac$ ...(3)
and given $\quad m^2 > n$ ...(4)
$\therefore$ $b^2 m^2 \geq anc \Rightarrow 4b^2m^2 - 4\,anc \geq 0 \Rightarrow D \geq 0$ {from (2)}
Hence roots of equation $ax^2 + 2mbx + nc = 0$ are real.

Ex.18 Show that the expression $x^2 + 2(a + b + c)\, x + 3(bc + ca + ab)$ will be a perfect square if $a = b = c$.

Sol. Given quadratic expression will be a perfect square if the discriminant of its corresponding equation is zero.
i.e. $4(a + b + c)^2 - 4.3\,(bc + ca + ab) = 0$ or $\quad (a + b + c)^2 - 3\,(bc + ca + ab) = 0$

or $\frac{1}{2}\left((a - b)^2 + (b - c)^2 + (c - a)^2\right) = 0$ which is possible only when $a = b = c$.

Ex.19 If $c < 0$ and $ax^2 + bx + c = 0$ does not have any real roots then prove that
(i) $a - b + c < 0$ **(ii)** $9a + 3b + c < 0$.

Sol. $c < 0$ and $D < 0$ $\Rightarrow$ $f(x) = ax^2 + bx + c < 0$ for all $x \in R$
and $f(3) = 9a + 3b + c < 0$ $\Rightarrow$ $f(-1) = a - b + c < 0$

G. EQUATION V/S IDENTITY

A quadratic equation is satisfied by exactly two values of 'x' which may be real or imaginary. The equation, $a x^2 + b x + c = 0$ is :

a quadratic equation if	$a \neq 0$	Two Roots
a linear equation if	$a = 0, b \neq 0$	One Root
a contradiction if	$a = b = 0, c \neq 0$	No Root
an identity if	$a = b = c = 0$	Infinite Roots

If a quadratic equation is satisfied by three distinct values of 'x', then it is an identity.
$(x + 1)^2 = x^2 + 2x + 1$ is an identity in x.
Here highest power of x in the given relation is 2 and this relation is satisfied by three different values $x = 0$, $x = 1$ and $x = -1$ and hence it is an identity because a polynomial equation of n^{th} degree cannot have more than n distinct roots.

Ex.20 If $a + b + c = 0$; $an^2 + bn + c = 0$ and $a + bn + cn^2 = 0$ where $n \neq 0, 1$, then prove that $a = b = c = 0$.

Sol. Note that $a x^2 + b x + c = 0$ is satisfied by $x = 1$; $x = n$ & $x = \frac{1}{n}$ where $n \neq \frac{1}{n}$
$\Rightarrow$ Q.E. has 3 distinct real roots which implies that it must be an identity.

Ex.21 If $\tan\alpha$, $\tan\beta$ are the roots of $x^2 - px + q = 0$ and $\cot\alpha$, $\cot\beta$ are the roots of $x^2 - rx + s = 0$ then find the value of *rs* in terms of *p* and *q*.

Sol. $x^2 - px + q = 0 \begin{cases} \tan\alpha \\ \tan\beta \end{cases}$ .....(1) ; $x^2 - rx + s = 0 \begin{cases} \cot\alpha \\ \cot\beta \end{cases}$ ....(2)
hence roots of 2^{nd} are reciprocal of (1)

$\therefore$ put $x \to \frac{1}{x}$ in (2) $\frac{1}{x^2} - \frac{r}{x} + s = 0$ $sx^2 - rx + 1 = 0$ ....(3)

comparing (1) and (3), $\frac{1}{s} = \frac{p}{r} = q$ $\Rightarrow s = \frac{1}{q}$; $r = \frac{p}{q}$ hence $rs = \frac{p}{q^2}$

H. SOLUTION OF QUADRATIC INEQUALITIES

The values of 'x' satisfying the inequality, $ax^2 + bx + c > 0$ $(a \neq 0)$ are :

(i) If $D > 0$, i.e. the equation $ax^2 + bx + c = 0$ has two different roots $\alpha < \beta$.
Then $a > 0 \Rightarrow$ $x \in (\infty, \alpha) \cup (\beta, \infty)$
$a < 0 \Rightarrow$ $x \in (\alpha, \beta)$

(ii) If $D = 0$, i.e. roots are equal, i.e. $\alpha = \beta$.
Then $a > 0 \Rightarrow$ $x \in (-\infty, \alpha) \cup (\alpha, \infty)$
$a < 0 \Rightarrow$ $x \in \phi$

(iii) If $D < 0$, i.e. the equation $ax^2 + bx + c = 0$ has no real roots.
Then $a > 0 \Rightarrow$ $x \in R$
$a < 0 \Rightarrow$ $x \in \phi$

(iv) Inequalities of the form $\frac{P(x)}{Q(x)} \begin{matrix} < \\ = \\ > \end{matrix} 0$ can be solved using the method of intervals.

Ex.22 Find the solution set of k so that $y = kx$ is secant to the curve $y = x^2 + k$.

Sol. put $y = kx$ in $y = x^2 + k \Rightarrow$ $kx = x^2 + k = 0 \Rightarrow x^2 - kx + k = 0$ for line to be secant, $D > 0$

$\Rightarrow k^2 - 4k > 0$ $k(k-4) > 0$

hence $k > 4$ or $k < 0 \Rightarrow k \in (-\infty, 0) \cup (4, \infty)$

Ex.23 Find out the values of 'a' for which any solution of the inequality, $\dfrac{\log_3(x^2 - 3x + 7)}{\log_3(3x+2)} < 1$ is also a solution of the inequality, $x^2 + (5 - 2a)x \le 10a$.

Sol. Note that $x^2 - 3x + 7 > 0 \quad \forall \quad x \in R$

Also $x > -2/3$ and $x \neq -1/3$

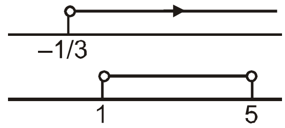
Also $3x + 2 > 1 \Rightarrow x > -1/3$ and $0 < 3x + 2 < 1 \Rightarrow -2/3 < x < -1/3$

Hence for $x > -1/3$

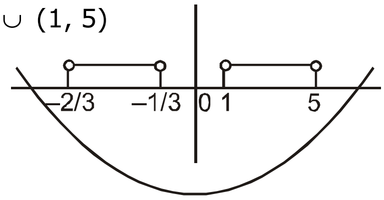
$\log_{3x+2}(x^2 - 3x + 7) < 1 \Rightarrow x^2 - 3x + 7 < 3x + 2$

$\Rightarrow x^2 - 6x + 5 < 0 \Rightarrow (x-5)(x-1) < 0 \quad \Rightarrow \quad x \in (1, 5)$

for $-2/3 < x < -1/3 \Rightarrow \quad x \in \left(-\dfrac{2}{3}, -\dfrac{1}{3}\right)$



Hence the solution of the first inequality is $x \in \left(-\dfrac{2}{3}, -\dfrac{1}{3}\right) \cup (1, 5)$



Now if any solution of the inequality is also the solution of,

$f(x) = x^2 + (5 - 2a)x - 10a \le 0$ then

$f(5) \le 0$ and $f\left(-\dfrac{2}{3}\right) \le 0 \quad \Rightarrow \quad a \ge 5/2$

Ex.24 Find the set of values of 'y' for which the inequality, $2\log_{0.5} y^2 - 3 + 2x \log_{0.5} y^2 - x^2 > 0$ is valid for atleast one real value of 'x'.

Sol. $-4\log_2 |y| - 3 - 4\log_2 |y|\, x - x^2 > 0$

or $x^2 + 4\log_2 |y|\, x + 3 + 4\log_2 |y| < 0$ or $x^2 + 4tx + 3 + 4t < 0$

for this inequality to be true for atleast one x, $3 > 0 \Rightarrow 4t^2 - 4t - 3 > 0$ $(2t - 3)(2t + 1) > 0$

$\Rightarrow t > \dfrac{3}{2}$ or $t < -\dfrac{1}{2}$ i.e. $\log_2 |y| > \dfrac{3}{2} \Rightarrow |y| > 2\sqrt{2} \Rightarrow y > 2\sqrt{2}$ or $y < 2\sqrt{2}$

Similarly $\log_2 |y| < -\dfrac{1}{2} \Rightarrow |y| < \dfrac{1}{\sqrt{2}} \Rightarrow y < \dfrac{1}{\sqrt{2}}$ or $y > -\dfrac{1}{\sqrt{2}}$

Ans. : $\left(-\infty, -2\sqrt{2}\right) \cup \left(-\dfrac{1}{\sqrt{2}}, 0\right) \cup \left(0, \dfrac{1}{\sqrt{2}}\right) \cup \left(2\sqrt{2}, \infty\right)$

Ex.25 Find the values of 'p' for which the inequality,

$\left(2 - \log_2\left(\dfrac{p}{p+1}\right)\right)x^2 + 2x\left(1 + \log_2\dfrac{p}{p+1}\right) - 2\left(1 + \log_2\dfrac{p}{p+1}\right) > 0$ is valid for all real x.

Sol. $(2 - t)x^2 + 2(1 + t)x - 2(1 + t) > 0$ when $t = 2$, $6x - 6 > 0$ which is not true $\forall \; x \in R$.

Let $t \neq 2$; $t < 2$ ......(1) and $4(1 + t)^2 + 8(1 + t)(2 - t) < 0$ (for given inequality to be valid)

or $(t - 5)(t + 1) > 0 \quad \Rightarrow \quad t > 5$ or $t < -1$ ......(2)

From (1) and (2) ; $t < -1 \Rightarrow \log_2 \dfrac{p}{p+1} < -1 \Rightarrow \dfrac{p}{p+1} < \dfrac{1}{2}$ or $\dfrac{p-1}{p+1} < 0 \Rightarrow -1 < p < 1$

but $\dfrac{p}{p+1} > 0 \quad \Rightarrow \quad p > 0$ or $p < -1 \quad \therefore$ common solution is $p \in (0, 1)$

I. RANGE OF QUADRATIC EXPRESSION $f(x) = ax^2 + bx + c$

(i) Range when $x \in R$: If $a > 0 \Rightarrow f(x) \in \left[-\frac{D}{4a}, \infty\right)$

$a < 0 \Rightarrow f(x) \in \left(-\infty, \frac{D}{4a}\right]$

Maximum & Minimum Value of $y = ax^2 + bx + c$ occurs at $x = -(b/2a)$ according as $a < 0$ or $a > 0$ respectively

(ii) Range in restricted domain : Given $x \in [x_1, x_2]$

(a) If $-\frac{b}{2a} \notin [x_1, x_2]$ then, $f(x) \in [\min\{f(x_1), f(x_2)\}, \max\{f(x_1), f(x_2)\}]$

(b) If $-\frac{b}{2a} \in [x_1, x_2]$ then, $f(x) \in \left[\min\left\{f(x_1), f(x_2), -\frac{D}{4a}\right\}, \max\left\{f(x_1), f(x_2), -\frac{D}{4a}\right\}\right]$

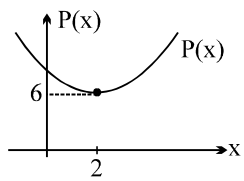
Ex.26 Find the minimum value of $f(x) = x^2 - 5x + 6$.

Sol. Minimum of $f(x) = -\frac{D}{4a} = -\left(\frac{25-24}{4}\right) = -\frac{1}{4}$ $\left(\text{at } x = -\frac{b}{2a} = \frac{5}{2}\right)$. Hence range is $\left[-\frac{1}{4}, \infty\right)$.

Ex.27 Let $P(x) = ax^2 + bx + 8$ is a quadratic polynomial. If the minimum value of $P(x)$ is 6 when $x = 2$, find the values of a and b.

Sol. $P(x) = ax^2 + bx + 8$ ....(1)

$P(2) = 4a + 2b + 8 = 6$ ....(2)



$\therefore\ -\frac{b}{2a} = 2;\ \therefore\ 4a = -b$

from (2), we get $-b + 2b = -2 \Rightarrow b = -2$ $\therefore$ $4a = -(-2)$ $\Rightarrow$ $a = 1/2$

Ex.28 If $\min(x^2 + (a-b)x + (1-a-b)) > \max(-x^2 + (a+b)x - (1+a+b))$, prove that $a^2 + b^2 < 4$.

Sol. Let $f(x) = x^2 + (a-b)x + (1-a-b) \Rightarrow f(x)_{min} = -\frac{D_1}{4}$, where D_1 is the discriminant of $f(x)$.

Let $g(x) = -x^2 + (a+b)x - (1+a+b) \Rightarrow g(x)_{max} = \frac{-D_2}{-4}$ where D_2 is the discriminant of $g(x)$.

Thus $-\frac{(a-b)^2 - 4(1-a-b)}{4} > -\frac{(a+b)^2 - 4(1+a+b)}{-4}$

$\Rightarrow 4(1-a-b) - (a-b)^2 > (a+b)^2 > (a+b)^2 - 4(1+a+b)$ or $8 > 2(a^2 + b^2) \Rightarrow a^2 + b^2 < 4$.

Ex.29 The coefficient of the quadratic equation $ax^2 + (a+d)x + (a+2d) = 0$ are consecutive terms of a positively valued, increasing arithmetic sequence. Determine the least integral value of d/a such that the equation has real solutions.

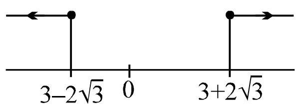
Sol. $ax^2 + (a + d)x + (a + 2d) = 0$

$a, \ a + d, \ a + 2d$ are in $\uparrow$ A.P. $(d > 0)$(note that positively valued terms $\Rightarrow a > 0$)

for real roots $D \geq 0 \Rightarrow (a + d)^2 - 4a(a + 2d) \geq 0 \Rightarrow \quad a^2 + d^2 + 2ad - 4a^2 - 8ad \geq 0$

$\Rightarrow \ (d - 3a)^2 - 12a^2 \geq 0 \quad \Rightarrow \quad (d - 3a - \sqrt{12}\,a)(d - 3a + \sqrt{12}\,a) \geq 0$

$\Rightarrow \ \left[\frac{d}{a} - (3 + 2\sqrt{3})\right]\left[\frac{d}{a} - (3 - 2\sqrt{3})\right] \geq 0$



$\therefore \ \left.\frac{d}{a}\right|_{Min} = 3 + 2\sqrt{3} \geq 6 \quad \Rightarrow \quad$ least integral value = 7

Ex.30 Let 'a' be real and α, β are the roots of the equation, $x^2 - 2ax + 1 = 0$. Find the value of 'a' for which $\frac{\alpha}{\beta^3} + \frac{\beta}{\alpha^3}$ attains the minimum value . Find the minimum value and the roots.

Sol. $\alpha + \beta = +2a \ ; \ \alpha\beta = 1$

$$E = \frac{\alpha^4 + \beta^4}{\alpha^3 \beta^3} = \frac{(\alpha^2 + \beta^2)^2 - 2\alpha^2\beta^2}{\alpha^3\beta^3} = \left[(\alpha + \beta)^2 - 2\alpha\beta\right]^2 - 2 = (4a^2 - 2)^2 - 2$$

$a = \pm \frac{1}{\sqrt{2}} \ ; \ \text{Min} = -2 \ ; \ \pm \frac{1}{\sqrt{2}}(1 \pm i)$

Ex.31 Find the greatest value of $\dfrac{\left\{\left(x + \frac{1}{x}\right)^4 - \left(x^4 + \frac{1}{x^4}\right) - 1\right\}}{\left(x + \frac{1}{x}\right)^2 + \left(x^2 + \frac{1}{x^2}\right)}$ for $\forall\, x \in R - \{0\}$.

Sol. Let $y = \dfrac{\left(x + \frac{1}{x}\right)^4 - \left(x^4 + \frac{1}{x^4}\right) - 1}{\left(x + \frac{1}{x}\right)^2 + \left(x^2 + \frac{1}{x^2}\right)}$.

Put $x + \frac{1}{x} = t \Rightarrow \quad y = \dfrac{t^4 - [(t^2 - 2)^2 - 2] - 1}{t^2 + t^2 - 2} = \dfrac{t^4 - [t^4 - 4t^2 + 2] - 1}{2(t^2 - 1)}$

$\Rightarrow \ y = \dfrac{4t^2 - 3}{2(t^2 - 1)} \Rightarrow y = \dfrac{4t^2 - 4 + 1}{2(t^2 - 1)} = \left(2 + \dfrac{1}{2(t^2 - 1)}\right)$.

As $t = x + \frac{1}{x} \Rightarrow t^2 \geq 4 \Rightarrow t^2 - 1 \geq 3 \Rightarrow \frac{1}{t^2 - 1} \leq \frac{1}{3} \Rightarrow$ Maximum value of y is $2 + \frac{1}{3 \times 2} = \frac{13}{6}$

Ex.32 The set of real parameter 'a' for which the equation $x^4 - 2ax^2 + x + a^2 - a = 0$ has all real solutions, is given by $\left[\frac{m}{n}, \infty\right)$ where *m* and *n* are relatively prime positive integers, find the value of (*m* + *n*).

Sol. We have $a^2 - (2x^2 + 1)a + x^4 + x = 0$

$\therefore a = \dfrac{(2x^2+1) \pm \sqrt{(2x^2+1)^2 - 4(x^4+x)}}{2} \Rightarrow 2a = (2x^2 + 1) \pm \sqrt{4x^2 - 4x + 1} = (2x^2 + 1) \pm (2x - 1)$

+ ve sign $\Rightarrow a = x^2 + x \quad \Rightarrow$ If $x^2 + x - a = 0 \Rightarrow x = \dfrac{-1 \pm \sqrt{1+4a}}{2}$

– ve sign $\Rightarrow 2a = 2x^2 - 2x + 2 \Rightarrow a = x^2 - x + 1$ If $x^2 - x + 1 - a \Rightarrow x = \dfrac{1 \pm \sqrt{1-4+4a}}{2} = \dfrac{1 \pm \sqrt{4a-3}}{2}$

for x to be real $a \geq 3/4$ and $a \geq -1/4 \Rightarrow a \geq 3/4 \Rightarrow 3 + 4 = 7$

Ex.33 Find the range of the expression $y = \dfrac{x^2 - 2x - 8}{x^2 - 4x - 5}$, for all permissible value of x.

Sol. $x^2y - 4xy - 5y = x^2 - 2x - 8 \Rightarrow (y - 1)x^2 + 2x(1 - 2y) + 8 - 5y = 0$

$\therefore$ $x \in R$ hence $D \geq 0 \Rightarrow 4(1 - 2y)^2 - 4(y - 1)(8 - 5y) \geq 0$

$\Rightarrow (4y^2 - 4y + 1) - (13y - 8 - 5y^2) \geq 0 \Rightarrow 9y^2 - 17y + 9 \geq 0$ ....(1)

since coefficient of $y^2 > 0$ and $D = 289 - 324 < 0$

Hence (1) is always true. Hence range of y is $(-\infty, \infty)$

Ex.34 Prove that if x is real, the minimum value of $\dfrac{(a+x)(b+x)}{(c+x)}$ $(x > -c)$, for $x > c$, $b > c$ is $(\sqrt{(a-c)} + \sqrt{(b-c)})^2$.

Sol. Let $y = \dfrac{(a+x)(b+x)}{(c+x)} \Rightarrow x^2 + (a + b)x + ab = cy + xy \Rightarrow x^2 + (a + b - y)x + ab - cy = 0$

for real x, $B^2 - 4AC \geq 0$

$\Rightarrow (a + b - y)^2 - 4ab + 4cy \geq 0 \Rightarrow (a + b)^2 + y^2 - 2(a + b)y - 4ab + 4cy \geq 0$

$\Rightarrow (a - b)^2 + y^2 - 2(a + b - 2c)y \geq 0 \Rightarrow y^2 - 2(a + b - 2c)y + (a - b)^2 \geq 0$

$\Rightarrow [y - (\sqrt{(a-c)} - \sqrt{(b-c)})^2][y - (\sqrt{(a-c)} + \sqrt{(b-c)})^2] \geq 0$

$\therefore y \leq (\sqrt{(a-c)} - \sqrt{(b-c)})^2$ and $y \geq (\sqrt{(a-c)} + \sqrt{(b-c)})^2$

Hence minimum value of y is $(\sqrt{(a-c)} + \sqrt{(b-c)})^2$.

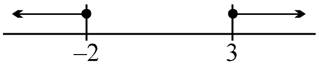
J. RESOLUTION OF A SECOND DEGREE EXPRESSION IN X AND Y

The condition that a quadratic function $f(x, y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c$ may be resolved into two linear factors is that $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ **OR** $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$.

Ex.35 If x is real and $4y^2 + 4xy + x + 6 = 0$, then find the complete set of values of x for which y is real.

Sol. $4y^2 + 4xy + x + 6 = 0 \quad y \in R$

$\therefore \quad D \geq 0 \Rightarrow \quad 16x^2 - 16(x + 6) \geq 0 \quad \Rightarrow \quad x^2 - x - 6 \geq 0 \quad \Rightarrow \quad (x - 3)(x + 2) \geq 0$



$\therefore \quad x \in (-\infty, -2] \cup [3, \infty)$

Ex.36 Find the greatest and the least real values of x & y satisfying the relation, $x^2 + y^2 = 6x - 8y$.

Sol. writing as a quadratic in y , $y^2 + 8y + x^2 - 6x = 0$

$y \in R \Rightarrow D \geq 0 \Rightarrow \quad x^2 - 6x - 16 \leq 0 \quad \Rightarrow -2 \leq x \leq 8$

Similarly range of y can also be obtained Ans. : $-2 \leq x \leq 8$ & $-9 \leq y \leq 1$

Ex.37 If x, y ans z are three real numbers such that x + y +z = 4 and $x^2 + y^2 + z^2 = 6$, then show that each of x, y and z lie in the closd interval $\left[\frac{2}{3}, 2\right]$.

Sol. Here, $\quad x^2 + y^2 + (4 - x - y)^2 = 6 \quad \Rightarrow \quad x^2 + (y - 4)x + (y^2 + 5 - 4y) = 0$

Since x is real $\Rightarrow D \geq 0 \quad \Rightarrow \quad (y - 4)^2 - 4(y^2 - 4y + 5) \geq 0 \Rightarrow \quad -3y^2 + 8y - 4 \geq 0$

$\Rightarrow \quad -(3y - 2)(y - 2) \geq 0 \quad \Rightarrow \quad \frac{2}{3} \leq y \leq 2$

Similarly, we can show that x and z $\in \left[\frac{2}{3}, 2\right]$.

K. THEORY OF EQUATIONS

If $\alpha_1, \alpha_2, \alpha_3, ...\alpha_n$ are the roots of the equation ; $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_{n-1}x + a_n = 0$ where $a_0, a_1, ... a_n$ are all real & $a_0 \neq 0$ then,

$$\Sigma \alpha_1 = -\frac{a_1}{a_0}, \ \Sigma \alpha_1 \alpha_2 = +\frac{a_2}{a_0}, \ \Sigma \alpha_1 \alpha_2 \alpha_3 = -\frac{a_3}{a_0}, ..., \alpha_1 \alpha_2 \alpha_3 ....\alpha_n = (-1)^n \frac{a_n}{a_0}$$

Note :

(i) If α is a root of the equation f(x) = 0, then the polynomial f(x) is exactly divisible by $(x - \alpha)$ or $(x - \alpha)$ is a factor of f(x) and conversely.

(ii) Every equation of nth degree $(n \geq 1)$ has exactly n roots & if the equation has more than n roots, it is an identity.

(iii) If the coefficients of the equation f(x) = 0 are all real and $\alpha + i\beta$ is its root, then $\alpha - i\beta$ is also a root. i.e. **imaginary roots occur in conjugate pairs**.

(iv) If the coefficients in the equation are all rational & $\alpha + \sqrt{\beta}$ is one of its roots, then $\alpha - \sqrt{\beta}$ is also a root where $\alpha, \beta \in Q$ & β is not a perfect square.

(v) If there be any two real numbers 'a' & 'b' such that f(a) & f(b) are of opposite signs, then f(x) = 0 must have atleast one real root between 'a' and 'b'.

(vi) Every equation f(x) = 0 of degree odd has atleast one real root of a sign opposite to that of its last term.

Ex.38 If x = 1 and x = 2 are solutions of the equation $x^3 + ax^2 + bx + c = 0$ and a + b = 1, then find the value of b.

Sol. $a + b + c = -1 \Rightarrow c = -2$ & $8 + 4a + 2b + c = 0 \Rightarrow 4a + 2b = -6 \Rightarrow 2a + b = -3$

$\Rightarrow a = -4,\ b = 5$ hence $a = -4;\ b = 5;\ c = -2$

Ex.39 A polynomial in x of degree greater than 3 leaves the remainder 2, 1 and –1 when divided by (x – 1); (x + 2) & (x + 1) respectively. Find the remainder, if the polynomial is divided by, $(x^2 - 1)(x + 2)$.

Sol. $f(x) = Q_1(x - 1) + 2 = Q_2(x + 2) + 1 = Q_3(x + 1) - 1 \Rightarrow f(1) = 2$; $f(-2) = 1$; $f(-1) = -1$

again $f(x) = Q_r(x^2 - 1)(x + 2) + ax^2 + bx + c$

Hence $a + b + c = 2$; $4a - 2b + c = 1$ and $a - b + c = -1$

Ex.40 Obtain a polynomial of lowest degree with integral coefficient, whose one of the zeroes is $\sqrt{5} + \sqrt{2}$.

Sol. Let $P(x) = x - (\sqrt{5} + \sqrt{2}) = [(x - \sqrt{5}) - \sqrt{2}]$

Now following the method used in the pervious example, using the conjugate, we get

$P_1(x) = [(x - \sqrt{5}) - \sqrt{2}]\ [(x - \sqrt{5}) + \sqrt{2}] = (x^2 - 2\sqrt{5}x + 5) - 2 = (x^2 + 3 - 2\sqrt{5}x)$

$P_2(x) = [(x^2 + 3) - 2\sqrt{5}x]\ [(x^2 + 3) + 2\sqrt{5}x] = (x^2 + 3)^2 - 20x^2 = x^4 + 6x^2 + 9 - 20x^2 = x^4 - 14x^2 + 9$

$P(x) = ax^4 + 14ax^2 + 9a$, where $a \in Q$, $a \neq 0$.

The other zeroes of this polynomial are $\sqrt{5} - \sqrt{2}, -\sqrt{5} + \sqrt{2}, -\sqrt{5} - \sqrt{2}$.

Ex.41 Find the roots of the equation $x^4 + x^3 - 19x^2 - 49x - 30$, given that the roots are all rational numbers.

Sol. Since all the roots are rational because, they are the divisors of –30.

The divisors of –30 are ± 1, ± 2, ± 3, ± 5, ± 6, ± 10, ± 15, ± 30.

By remainder theorem, we find that –1, –2, – 3, and 5 are the roots.

Hence the roots are –1, –2, –3 and + 5.

Ex.42 From the equation of the lowest degree with rational co–efficients, which has $2 + \sqrt{3}$ and $3 + \sqrt{2}$, as two of its roots.

Sol. Since in an equation with rational co–efficients, irrational roots occur in pairs, therefore the required equation must have at least four roots, namely, $2 \pm \sqrt{3}$, $3 \pm \sqrt{2}$, i.e., it must have $\{x - (2 + \sqrt{3})\}\ \{x - (2 - \sqrt{3})\}\ \{x - (3 + \sqrt{2})\}\ \{x - (3 - \sqrt{2})\}$

as a factor, i.e., it must have $\{(x - 2)^2 - 3\}\ \{(x - 3)^2 - 2)\}$ ...(1)

Since the co–efficients of (1) are rational, therefore $\{(x - 2)^2 - 3\}\ \{(x - 3)^2 - 2\} = 0$ is required equation.

Ex.43 Let u, v be two real numbers such that u, v and uv are roots of a cubic polynomial with rational coefficients. Prove or disprove uv is rational.

Sol. Let $x^3 + ax^2 + bx + c = 0$ be the cubic polynomial of which u, v and uv are the roots and a, b, c are all rationals.

$u + v + uv = -a \Rightarrow u + v = -a - uv, uv + uv^2 + u^2v = b$ and $u^2v^2 = -c$

$\Rightarrow b = uv + uv^2 + u^2v = uv(1 + v + u) = uv(1 - a - uv) = (1 - a)uv - u^2v^2 = (1 - a)uv + c$

i.e., $uv = \dfrac{(b - c)}{1 - a}$ and since a, b, c are rational, uv is rational.

Ex.44 Prove that the equation $x^4 + 2x^3 + 5 + ax^3 + a = 0$ has at the most two real roots for all values of $a \in R - \{-5\}$.

Sol. The given expression is $(x^3 + 1)^2 + a(x^3 + 1) + 4 = 0$

If discriminant of the above equation is less than zero i.e. $D < 0$

Then we have six complex roots and no real roots.

If $D \geq 0$, $x^3 + 1 = t$, then the equation reduces to $f(t) = t^2 + at + 4 = 0$

we will get two real roots and other roots will be complex except when $t = 1$ is one of the roots

$\Rightarrow$ $f(1) = 0 \Rightarrow a = -5$.

Ex.45 The product of two of the four roots of $x^4 - 20x^3 + kx^2 + 590x - 1992 = 0$ is 24. Find k.

Sol. Let the given equation be written as $f(x) = 0$, and let the roots of the equation be r_1, r_2, r_3, r_4 with $r_1r_2 = 24$. Now $r_1r_2r_3r_4 = -1992$, so $r_3r_4 = -1992/24 = -83$, Also,
$f(x) = (x - r_1)(x - r_2)(x - r_3)(x - r_4) = (x^2 - cx + r_1r_2)(x^2 - dx + r_3r_4) = (x^2 - cx + 24)(x^2 - dx - 83)$, with $c = r_1 + r_2$, $d = r_3 + r_4$. Comparing coefficients of x^3 and x we get $c + d = 20$ and $83c - 24d = 590$. This gives $c = 10$, $d = 10$. Comparing coefficients of x^2, $k = cd - 83 + 24 = 100 - 83 + 24 = 41$.

Ex.46 If the equation, $ax^3 + 3bx^2 + 3cx + d = 0$ has two equal roots, show that each of them is equal to

$$\frac{bc - ad}{2(ac - b^2)}.$$

Sol. If α is a repeated root of $f(x) = ax^3 + 3bx^2 + 3cx + d = 0$

$\Rightarrow$ α is also a root of $f'(x) = 3ax^2 + 6bx + 3c = 0$

i.e. $a\alpha^2 + 2b\alpha + c = 0$ or $a\alpha^3 + 2b\alpha^2 + c\alpha = 0$ ....(1)

also $a\alpha^3 + 3b\alpha^2 + 3c\alpha + d = 0$

substituting $b\alpha^2 + 2c\alpha + d = 0$ ....(2)

From (1) and (2) $\alpha = \dfrac{bc - ad}{2(ac - b^2)}$

Ex.47 Let a, b, c be the three roots of the equation $x^3 + x^2 - 333x - 1002 = 0$ then find the value of $a^3 + b^3 + c^3$.

Sol. Let t be the root of the given cubic where t can take values a, b, c

hence $t^3 + t^2 - 333t - 1002 = 0$ or $t^3 = 1002 + 333t - t^2$

$\therefore$ $\sum t^3 = \sum 1002 + 333\sum t - \sum t^2 = 3006 + 333\sum t - \left[\left(\sum t\right)^2 - 2\sum t_1t_2\right]$

but $\sum t = -1$; $\sum t_1t_2 = -333$

$\therefore$ $a^3 + b^3 + c^3 = 3006 - 333 - [1 + 666] = 3006 - 333 - 667 = 3006 - 1000 = 2006$

Ex.48 If $\alpha, \beta, \gamma, \delta$ be the roots of the equation $x^4 + px^3 + qx^2 + rx + x = 0$,
show that $(1 + \alpha^2)(1 + \beta^2)(1 + \gamma^2)(1 + \delta^2) = (1 - q + s)^2 + (p - r)^2$.

Sol. Since $\alpha, \beta, \gamma, \delta$ are the roots of the equation $x^4 + px^3 + qr^2 + rx + x = 0$,

therfore $x^4 + px^3 + qx^2 + rx + x \equiv (x - \alpha)(x - \beta)(x - \gamma)(x - \delta)$,

Substituting $x = i, -i$ successively, we have

$(1 - q + s) - i(p - r) = (i - \alpha)(i - \beta)(i - \gamma)(i - \delta)$ ....(1)

$(1 - q + s) + i(p - r) = (-i - \alpha)(-i - \beta)(-i - \gamma)(-i - \delta)$ ....(2)

Multiplying corresponding sides of (1) and (2), we have

$(1 - q + s)^2 + (p - r)^2 = (1 + \alpha^2)(1 + \beta^2)(1 + \gamma^2)(1 + \delta^2)$.

Ex.49 Let α, β and γ are the roots of the cubic $x^3 - 3x^2 + 1 = 0$. Find a cubic whose roots are $\frac{\alpha}{\alpha-2}$, $\frac{\beta}{\beta-2}$ and $\frac{\gamma}{\gamma-2}$. Hence or otherwise find the value of $(\alpha - 2)(\beta - 2)(\gamma - 2)$.

Sol. Let $y = \frac{x}{x-2}$ where x can take values α, β, γ

$\therefore$ $xy - 2y = x \Rightarrow$ $x = \frac{2y}{y-1}$ ....(1)

substituting the value of x in $x^3 - 3x^2 + 1 = 0$ and simplifying we get the required cubic as

$3y^3 - 9y^2 - 3y + 1 = 0$ ....(2) Ans.

now from (2) $\frac{\alpha}{\alpha-2} \cdot \frac{\beta}{\beta-2} \cdot \frac{\gamma}{\gamma-2} = -\frac{1}{3}$ $\therefore$ $(\alpha - 2)(\beta - 2)(\gamma - 2) = -3\alpha\beta\gamma$

but $\alpha\gamma\beta = -1$ (given) $\therefore$ $(\alpha - 2)(\beta - 2)(\gamma - 2) = 3$

Ex.50 Let α and β be the roots of the quadratic equation $(x - 2)(x - 3)+(x - 3)(x + 1)+(x + 1)(x - 2)=0$. Find the value of $\frac{1}{(\alpha+1)(\beta+1)} + \frac{1}{(\alpha-2)(\beta-2)} + \frac{1}{(\alpha-3)(\beta-3)}$.

Sol. Quadratic equation is $3x^2 - 8x + 1 = 0 \begin{cases} \alpha \\ \beta \end{cases}$ $\Rightarrow$ $\alpha + \beta = \frac{8}{3}$ and $\alpha\beta = \frac{1}{3}$

the value of $\frac{1}{\alpha\beta+(\alpha+\beta)+1} + \frac{1}{\alpha\beta-2(\alpha+\beta)+4} + \frac{1}{\alpha\beta-3(\alpha+\beta)+9} = \frac{1}{4} - 1 + \frac{3}{4} = -\frac{3}{4} + \frac{3}{4} = 0$

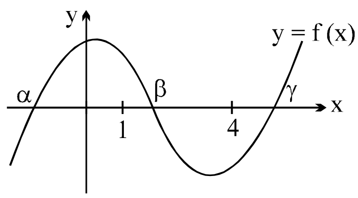
Ex.51 Given the cubic equation $x^3 - 2kx^2 - 4kx + k^2 = 0$. If one root of the equation is less than 1, other root is in the interval (1, 4) and the 3rd root is greater than 4, then the value of k lies in the interval $\left(a+\sqrt{b}, b(a+\sqrt{b})\right)$ where a, b $\in$ N. Find the value of $(a + b)^3 + (ab + 2)^2$.

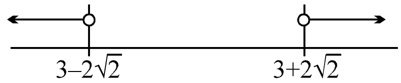
Sol. $f(x) = x^3 - 2kx^2 - 4kx + k^2 = 0$

note that $f(0) = k^2 > 0$

$f(1) > 0 \Rightarrow 1 - 2k - 4k + k^2 > 0 \Rightarrow k^2 - 6k + 1 > 0$

$\therefore$ $[k - (3+2\sqrt{2})]\,[k - (3-2\sqrt{2})] > 0$



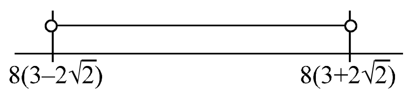


....(1)

Also $f(4) < 0 \Rightarrow$ $64 - 32k - 16k + k^2 < 0$ $\Rightarrow k^2 - 48k + 64 < 0$ $\Rightarrow (k - 24)^2 < 512$

$\Rightarrow$ $(k - 24 + 16\sqrt{2})(k - 24 - 16\sqrt{2}) < 0$

$\Rightarrow$ $[k - 8(3-2\sqrt{2})]\,[k - 8(3+2\sqrt{2})] < 0$



....(2)

(1) $\cap$ (2) $\Rightarrow$ $3+2\sqrt{2} < k < 8(3+2\sqrt{2})$ $\quad$ $3+\sqrt{8} < k < 8(3+\sqrt{8})$

$\therefore$ a = 3; b = 8 $\quad (a + b)^3 + (ab + 2)^2 \Rightarrow (11)^3 + (26)^2 = 1331 + 676 = 2007$

Ex.52 If p(x) is a polynomial with integer coefficients and a, b, c are three distinct integers, then show that it is impossible to have p(a) = b, p(b) = c and p(c) = a.

Sol. Suppose a, b, c are distinct integers such that p(a) = b, p(b) = c and p(c) = a. Then

$p(a) - p(b) = b - c, \quad p(b) - p(c) = c - a, \quad p(c) - p(a) = a - b.$

But for any two integers $m \neq n$, $m - n$ divides $p(m) - p(n)$. Thus we get,

$a - b \mid b - c, \quad b - c \mid c - a, \quad c - a \mid a - b.$

Therefore a = b = c, a contradiction, Hence there are no integers a, b, and c such that p(a) = b, p(b) = c and p(c) = a.

Ex.53 Find all cubic polynomials p(x) such that $(x - 1)^2$ is a factor of p(x) + 2 and $(x + 1)^2$ is a factor of p(x) – 2.

Sol. If $(x - \alpha)$ divides a polynomial q(x) then $q(\alpha) = 0$. Let $p(x) = ax^3 + bx^2 + cx + d$. Since (x – 1) divides p(x) + 2, we get $a + b + c + d + 2 = 0$.

Hence $d = -a - b - c - 2$ and

$p(x) + 2 = a(x^3 - 1) + b(x^2 - 1) + c(x - 1) = (x - 1)\{a(x^2 + x + 1) + b(x + 1) + c\}$.

Since $(x - 1)^2$ divides p(x) + 2, we conclude that (x – 1) divides $a(x^2 + x + 1) + b(x + 1)$ c. This implies that 3a + 2b + c = 0. Similarly, using the information that $(x + 1)^2$ divides p(x) – 2, we get two more relations : –a + b – c + d – 2 = 0; 3a – 2b + c = 0. Solving these for a, b, c, d, we obtain b = d = 0, and a = 1, c = –3. Thus there is only one polynomial satisfying the given condition : $p(x) = x^3 - 3x$.

Ex.54 Suppose a, b, c are rational numbers and all the roots of $x^4 - ax^2 + bx + c = 0$ are rational and distinct. Let p, q, r, s be these roots. Prove that the number $4(a + pq) - 3(p + q)^2$ is the square of a rational number.

Sol. Using the relations between roots and coefficients we have

p + q + r + s = 0 or p +q = –(r + s), pq + pr + ps + qr + qs + rs = –a.

Hence subsituting for a and p +q,

$$4(a + pq) - 3(p + q)^2 = 4[-(pq + pr + ps + qr + qs + rs) + pq] - 3(r + s)^2$$
$$= 4[-(p + q)(r + s) - rs] - 3(r + s)^2 = 4(r + s)^2 - 4rs - 3(r + s)^2 = (r - s)^2,$$

which is the square of the rational number r – s.

L. COMMON ROOTS OF QUADRATIC EQUATIONS

Atleast one Common Root :

Let α be the common root of $ax^2 + bx + c = 0$ & $a'x^2 + b'x + c' = 0$.

Therefore $a\alpha^2 + b\alpha + c = 0$; $a'\alpha^2 + b'\alpha + c' = 0$.

By Cramer's Rule $\dfrac{\alpha^2}{bc' - b'c} = \dfrac{\alpha}{a'c - ac'} = \dfrac{1}{ab' - a'b}$. Therefore, $\alpha = \dfrac{ca' - c'a}{ab' - a'b}$, $\alpha^2 = \dfrac{bc' - b'c}{ab' - a'b}$

So the condition for a common root is $(ca' - c'a)^2 = (ab' - a'b)(bc' - b'c)$.

Note : If f(x) = 0 & g(x) = 0 are two polynomial equation having some common roots(s) then those common root(s) is/are also the root(s) of h(x) = a f(x) + bg (x) = 0.

Ex.55 If a, b, c $\in$ R and equations $ax^2 + bx + c = 0$ and $x^2 + 2x + 9 = 0$ have a common root, show that a : b : c = 1 : 2 : 9.

Sol. Given equations are : $x^2 + 2x + 9 = 0$ .... (i) and $ax^2 + bx + c = 0$ .... (ii)

Clearly roots of equation (i) are imgainary since equation (i) and (ii) have a common root, therefore common root must be imaginary and hence both roots will be common.

Threfore equatons (i) and (ii) are identical $\therefore$ $\frac{a}{1} = \frac{b}{2} = \frac{c}{9}$ $\therefore$ a : b : c = 1 : 2 : 9

Ex.56 Determine a such that $x^2 - 11x + a$ and $x^2 - 14x + 2a$ may have a common factor.

Sol. Let $x - \alpha$ be a common factor of $x^2 - 11x + a$ and $x^2 - 14x + 2a$.

Then $x = \alpha$ will satisfy the equations $x^2 - 11x + a = 0$ and $x^2 - 14x + 2a = 0$

$\therefore$ $\alpha^2 - 11\alpha + a = 0$ and $\alpha^2 - 14\alpha + 2a = 0$

Solving (i) and (ii) by cross multiplication method, we get a = 24.

Ex.57 If the quadratic equations, $x^2 + bx + c = 0$ and $bx^2 + cx + 1 = 0$ have a common root then prove that either $b + c + 1 = 0$ or $b^2 + c^2 + 1 = bc + b + c$.

Sol. $\alpha^2 + b\alpha + c = 0$ ....(1) $b\alpha^2 + c\alpha + 1 = 0$ ....(2)

$$\therefore \quad \frac{\alpha^2}{b - c^2} = \frac{\alpha}{bc - 1} = \frac{1}{c - b^2} \quad \Rightarrow \quad \alpha^2 = \frac{b - c^2}{c - b^2} \text{ or } \alpha = \frac{bc - 1}{c - b^2}$$

$\Rightarrow$ $(bc - 1)^2 = (b - c^2)(c - b^2)$ or $b^3 + c^3 + 1 - 3bc = 0$

$\Rightarrow$ $(b + c + 1)(b^2 + c^2 + 1 - bc - c - b) = 0$ $\Rightarrow$ $b + c + 1 = 0$ or $b^2 + c^2 + 1 = bc + b + c$

Ex.58 Determine all possible value(s) of 'p' for which the equation $ax^2 - px + ab = 0$ and $x^2 - ax - bx + ab = 0$ may have a common root, given a, b are non zero real numbers.

Sol. $x^2 - (a + b)x + ab = 0$ or $(x - a)(x - b) = 0$ $\Rightarrow$ x = a or b

If x = a is the root of other equation, $a^3 - ap + ab = 0$ $\Rightarrow$ $p = a(a + b)$

If x = b is the root of the other equation, then $ab^2 - pb + ab = 0$ $\Rightarrow$ $p = a(1 + b)$

Ans. $p = a(a + b)$ or $a(1 + b)$

Ex.59 If $x^2 - ax + b = 0$ and $x^2 - px + q = 0$ have a root in common and the second equation has equal roots show that $b + q = \frac{ap}{2}$.

Sol. Given equation are $x^2 - ax + b = 0$ ans $x^2 - px + q = 0$. Let α be the common root. Then roots of equation (2) will be α and α. Let β be the other root of equation (1). Thus roots of equation (1) are α, β and those of equation (2) are α, α.

Now $\alpha + \beta = a$ .... (iii), $\alpha\beta = b$ .... (iv), $2\alpha = p$ .... (v), $\alpha^2 = q$ .... (vi)

L.H.S. = $b + q = \alpha\beta + \alpha^2 = \alpha(\alpha + \beta)$ .... (vii) and R.H.S. = $\frac{ap}{2} = \frac{(\alpha + \beta)2\alpha}{2} = \alpha(\alpha + \beta)$ .... (viii)

from (7) and (8), L.H.S. = R.H.S.

Ex.60 If each pair of the following three equations $x^2 + ax + b = 0$, $x^2 + cx + d = 0$, $x^2 + ex + f = 0$ has exactly one root in common, then show that $(a + c + e)^2 = 4(ac + ce + ea - b - d - f)$

Sol. $x^2 + ax + b = 0$ ...(1) $x^2 + cx + d = 0$ ...(2) $x^2 + ex + f = 0$ ...(3)

Let α, β be the roots of (1), β, γ be the roots of (2) and γ, δ be the roots of (3).

$\therefore$ $\alpha + \beta = -a$, $\alpha\beta = b$ ...(4), $\beta + \gamma = -c$, $\beta\gamma = d$ ...(5), $\gamma + \alpha = -e$, $\gamma\alpha = f$ ...(6)

$\therefore$ L.H.S. $= (a + c + e)^2 = (-\alpha - \beta - \beta - \gamma - \gamma - \alpha)^2 = 4(\alpha + \beta + \gamma)^2$ ...(7) {from (4), (5) & (6)}

R.H.S. $= 4(ac + ce + ea - b - d - f)$

$= 4\{(\alpha + \beta)(\beta + \gamma) + (\beta + \gamma)(\gamma + \alpha) + (\gamma + \alpha)(\alpha + \beta) - \alpha\beta - \beta\gamma - \gamma\alpha)\}$ {from (4), (5) & (6)}

$= 4(\alpha^2 + \beta^2 + \gamma^2 + 2\alpha\beta + 2\beta\gamma + 2\gamma\alpha) = (\alpha + \beta + \gamma)^2$ ...(8)

From (7) & (8), $(a + c + e)^2 = 4(ac + ce + ea - b - d - f)$

Ex.61 Find the common roots of $x^4 + 5x^3 - 2x^2 - 50x + 132 = 0$ and $x^4 + x^3 - 20x^2 + 16x + 24 = 0$ and hence solve the equations.

Sol. We can see that $4(x^2 - 5x + 6)$ is H.C.F. of the two equations and hence, the common roots are the roots of $x^2 - 5x + 6 = 0$ i.e., $x = 3$ or $x = 2$.

Now, $x^4 + 5x^3 - 22x^2 - 50x + 132 = 0$ ....(1) and $x^4 + x^3 - 20x^2 + 16x + 24 = 0$ ....(2)

have 2 and 3 as their common roots.

Ex.62 Suppose a cubic polynomial $f(x) = x^3 + px^2 + qx + 72$ is divisible by both $x^2 + ax + b$ and $x^2 + bx + a$ (where a, b, p, q are constants and $a \neq b$). Find the sum of the squares of the roots of the cubic polynomial.

Sol. Since cubic is divisible by both $x^2 + ax + b$ and $x^2 + bx + a$ and

$\therefore$ $x^2 + ax + b$ and $x^2 + bx + a$ must have a common roots.

$x^2 + bx + a = 0$ subtract from $x^2 + ax + b = 0$

$\Rightarrow$ $x(a - b) = (a - b)$ $\Rightarrow$ $x = 1$ $\therefore$ common root is 1

$x^2 + ax + b = 0 \begin{cases} 1 \\ \alpha \end{cases} \Rightarrow 1 \cdot \alpha = b \Rightarrow \alpha = b$ and $x^2 + bx + a = 0 \begin{cases} 1 \\ \beta \end{cases} \Rightarrow 1 \cdot \beta = a \Rightarrow \beta = a$

$\Rightarrow$ roots of cubic be 1, a, b

product of the roots be $1 \cdot a \cdot b = -72$ ...(1) and $a + b + 1 = 0$ ...(2) (from $x^2 + ax + b = 0$ put $x = 1$)

$\therefore$ $a - \dfrac{72}{b} = -1$ $\Rightarrow$ $a^2 + a - 72 = 0$ $\Rightarrow$ $(a + 9)(a - 8) = 0$ $\Rightarrow$ $a = -9, 8$

$\therefore$ roots are $1, -9, 8$ $\Rightarrow$ sum of their squares $= 1 + 81 + 64 = 146$

M. LOCATION OF ROOTS

Let $f(x) = ax^2 + bx + c$, where $a > 0$ & $a, b, c \in R$.

(i) Conditions for both the roots of $f(x) = 0$ to be greater than a specified number 'd' are $b^2 - 4ac \geq 0$; $f(d) > 0$ & $(-b/2a) > d$.

(ii) Conditions for both roots of $f(x) = 0$ to lie on either side of the number 'd' (in other words the number 'd' lies between the roots of $f(x) = 0$ is $f(d) < 0$.

(iii) Conditions for exactly one root of $f(x) = 0$ to lie in the interval (d, e) i.e. $d < x < e$ are $b^2 - 4ac > 0$ & $f(d) \cdot f(e) < 0$.

(iv) Conditions that both roots of $f(x) = 0$ to be confined between the numbers p & q are $(p < q)$. $b^2 - 4ac \geq 0$; $f(p) > 0$; $f(q) > 0$ & $p < (-b/2a) < q$.

Ex.63 If α & β are the roots of $x^2 + px + q = 0$ and α^4, β^4 are the roots of $x^2 - rx + s = 0$, then the equation $x^2 - 4qx + 2q^2 - r = 0$ has always (p, q, r, s $\in$ R)

Sol. $\alpha + \beta = -p$; $\alpha\beta = q$; $\alpha^4 + \beta^4 = r$; $\alpha^4\beta^4 = s$

Now $D = 16q^2 - 4(2q^2 - r) = 8q^2 + 4r = 4(2q^2 + r) = 4[2\alpha^2\beta^2 + \alpha^4 + \beta^4]$

$= 4[(\alpha^2 + \beta^2)^2] = 4[(\alpha + \beta)^2 - 2\alpha\beta]^2 = 4(p^2 - 2q)^2 > 0$

Again consider the product of the roots $= 2q^2 - r$ which can be either positive or negative.

Ex.64 If the roots of the quadratic equation, $(a^2 - 7a + 13)x^2 - (a^3 - 8a - 1)x + \log_{1/2}(a^2 - 6a + 9) = 0$ lie on either side of origin, then find the range of value of 'a'.

Sol. $f(0) < 0 \Rightarrow \log_{1/2}(a-3)^2 < 0 \Rightarrow (a-3)^2 > 1 \Rightarrow (a-4)(a-2) > 0 \Rightarrow (-\infty, 2) \cup (4, \infty)$

Ex.65 Find the set of values of 'p' for which the quadratic equation, $(p - 5)x^2 - 2px - 4p = 0$ has atleast one positive root.

Sol. For real roots $D \geq 0 \Rightarrow p(p-4) \geq 0$; $p \neq 5 \Rightarrow (-\infty, 0] \cup [4, 5) \cup (5, \infty)$

For both non positive roots sum ≤ 0 ; product ≥ 0 & $D \geq 0 \Rightarrow [4, 5)$

Hence for atleast one positive root $p \in (-\infty, 0] \cup (5, \infty)$

Ex.66 Find the range of values of a for which the equation $x^2 - (a-5)x + \left(a - \frac{15}{4}\right) = 0$ has at least one positive root.

Sol. $D \geq 0 \Rightarrow (a-5)^2 - 4\left(a - \frac{15}{4}\right) \geq 0 \Rightarrow a^2 - 10a + 25 - 4a + 15 \geq 0$

$\Rightarrow a^2 - 14a + 40 \geq 0 \Rightarrow (a-4)(a-10) \geq 0 \Rightarrow a \in (-\infty, 4] \cup [10, \infty)$

Case I : When both roots are positive

$D \geq 0, a - 5 > 0, a - \frac{15}{4} > 0 \Rightarrow D \geq 0, a > 5, a > \frac{15}{4} \Rightarrow a \in [10, \infty)$

Case II : when exactly one root is positive $\Rightarrow a - \frac{15}{4} \leq 0 \Rightarrow a \leq \frac{15}{4}$

then finally $a \in (-\infty, 15/4] \cup [10, \infty)$

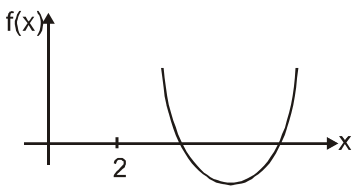
Ex.67 Consider the quadratic equation $x^2 - (m-3)x + m = 0$ and answer the questions that follow.

(a) Find values of m so that both the roots are greater than 2.

Condition–I $D \geq 0 \Rightarrow (m-3)^2 - 4m \geq 0 \Rightarrow m^2 - 10m + 9 \geq 0$

$\Rightarrow (m-1)(m-9) \geq 0 \Rightarrow m \in (-\infty, 1] \cup [9, \infty)$ ....(i)

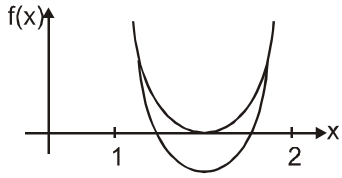
Condition – II $f(2) > 0 \Rightarrow 4 - (m-3)2 + m > 0 \Rightarrow m < 10$ ....(ii)



Condition – III $-\frac{b}{2a} > 2 \Rightarrow \frac{m-3}{2} > 2 \Rightarrow m > 7$ .... (iii)

Intersection of (i), (ii) and (iii) gives $m \in [9, 10)$

(b) Find the values of m so that both roots lie in the interval (1, 2)



Condition – I $D \geq 0 \Rightarrow m \in (-\infty. 1] \cup [9, \infty)$

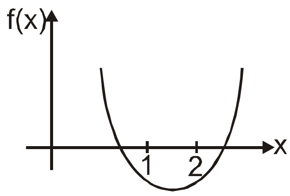
Condition – II $f(1) > 0 \Rightarrow 1 - (m - 3) + m > 0 \Rightarrow 4 > 0 \Rightarrow m \in R$

Condition – III $f(2) > 0 \Rightarrow m < 10$

Condition – IV $1 < -\dfrac{b}{2a} < 2 \Rightarrow 1 < \dfrac{m-3}{2} < 2 \Rightarrow 5 < m < 7$

Intersection gives $m \in \phi$

(c) One root is greater than 2 and other smaller than 1

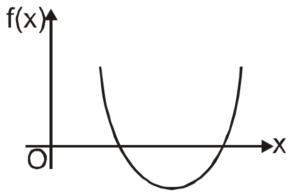


Condition – I $f(1) < 0 \Rightarrow 4 < 0 \Rightarrow m \in \phi$

Condition – II $f(2) < 0 \Rightarrow m < 10$

Intersection gives $m \in \phi$Ans.

(d) Find the value of m for which both roots are positive.



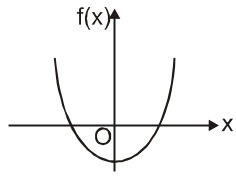
Condition – I $D \geq 0 \Rightarrow m \in (-\infty, 1] \cup [9, \infty)$

Condition – II $f(0) > 0 \Rightarrow m > 0$

Conditon – III $-\dfrac{b}{2a} > 0 \Rightarrow \dfrac{m-3}{2} > 0 \Rightarrow m > 3$

Intersection gives $m \in [9, \infty)$

(e) Find the values of m for which one root is (positive) and other is (negative).



Condition – I $f(0) < 0 \Rightarrow m < 0$

(f) Roots are equal in magnitude and opposite in sign.

Sum of roots = 0 $\Rightarrow$ m = 3 and $f(0) < 0 \Rightarrow m < 0 \quad \therefore \ m \in \phi$

Ex.68 Find all the values of 'a' for which both the roots of the equation.

$(a - 2)x^2 + 2ax + (a + 3) = 0$ lies in the interval (–2, 1).

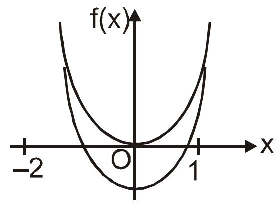
Sol. **Case – I** When $a - 2 > 0 \Rightarrow a > 2$

Conditon – I $f(-2) > 0 \Rightarrow (a - 2)\,4 - 4a + a + 3 > 0 \Rightarrow a - 5 > 0 \Rightarrow a > 5$

Conditon – II $f(1) > 0 \Rightarrow 4a + 1 > 0 \Rightarrow a > -\dfrac{1}{4}$

Conditon – III $D \geq 0 \Rightarrow 4a^2 - 4(a + 3)(a - 2) \geq 0 \Rightarrow a \leq 6$

Conditon – IV $-\dfrac{b}{2a} < 1 \Rightarrow \dfrac{2(a-1)}{a-2} > 0 \Rightarrow a \in (-\infty, 1) \cup (4, \infty)$



Conditon – V $-2 < -\dfrac{b}{2a} < 1 \Rightarrow \dfrac{-2a}{2(a-2)} > -2 \Rightarrow \dfrac{a-4}{a-2} > 0$

Intersection gives $a \in (5, 6]$

Case – II When $a - 2 < 0 \Rightarrow a < 2$

Condition – I $f(-2) < 0 \quad \Rightarrow \quad a < 5$

Condition – II $f(1) < 0, \quad \Rightarrow \quad a < -\frac{1}{4}$

Condition – III $-2 < -\frac{b}{2a} < 1 \quad \Rightarrow \quad a \in (-\infty, 1) \cup (4, \infty)$

Condition – IV $D \geq 0 \quad \Rightarrow \quad a \leq 6$

intersection gives $a \in \left(-\infty, -\frac{1}{4}\right)$. Hence complete solution is $a \in \left(-\infty, -\frac{1}{4}\right) \cup (5, 6]$

Ex.69 The coefficients of the equation $ax^2 + bx + c = 0$ where $a \neq 0$, satisfy the inequality $(a + b + c)(4a - 2b + c) < 0$. Prove that this equation has 2 distinct real solutions.

Sol. $ax^2 + bx + c = 0 \quad (a \neq 0) \quad \Rightarrow$ given $(a + b + c)(4a - 2b + c) < 0$

$f(1) = a + b + c \quad f(-2) = 4a - 2b + c \quad \Rightarrow f(1) \cdot f(-2) < 0$

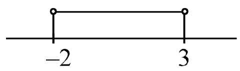
since $f(1)$ and $f(-2)$ have opposite signs

$\therefore$ exactly one root lies between -2 and 1

$\therefore$ this quadratic equation has 2 distinct real roots.

Ex.70 Find the value of k for which one root of the equation of $x^2 - (k + 1)x + k^2 + k - 8 = 0$ exceed 2 and other is smaller than 2.

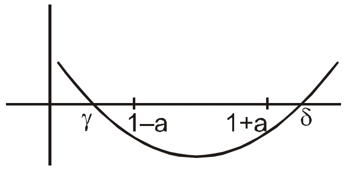
Sol. Since $a > 0$, $f(0) < 0 \quad \therefore \quad f(2) < 0$



$\therefore \quad 4 - 2(k + 1) + k^2 + k - 8 < 0 \quad \Rightarrow k^2 - K + 6 < 0 \quad \Rightarrow (k + 2)(k - 3) < 0 \ \therefore \ k \in (-2, 3)$

Ex.71 If α, β are the roots of the equation, $x^2 - 2x - a^2 + 1 = 0$ and γ, δ are the roots of the equation, $x^2 - 2(a + 1)x + a(a - 1) = 0$ such that $\alpha, \beta \in (\gamma, \delta)$ then find the values of 'a'.

Sol. $\alpha, \beta = \frac{2 \pm \sqrt{4 + 4(a^2 - 1)}}{2} = 1 + a$ or $1 - a$



Now $f(x) = x^2 - 2(a + 1)x + a(a - 1) = 0$

for $\alpha, \beta \in (\gamma, \delta)$, $f(1 + a) < 0$ and $f(1 - a) < 0 \quad \Rightarrow a \in \left(-\frac{1}{4}, 1\right)$

Ex.72 **(a)** For what values of 'a' exactly one root of the equation $2^a x^2 - 4^a x + 2^a - 1 = 0$, lies between a and 2.

(b) Find all values of a for which the equation $2x^2 - 2(2a + 1)x + a(a + 1) = 0$ has two roots, one of which is greater than a and the other is smaller than a.

Sol. **(a)** $\because$ Exactly one root of the given equation lies between 1 and 2 then $f(1) f(2) < 0$

Here $f(x) = 2^a x^2 - 4^a x + 2^a - 1$

$\therefore \quad (2^a - 4^a + 2^a - 1)(4.2^a - 2.4^a + 2^a - 1) < 0$

$\Rightarrow \quad (4^a - 2.2^a + 1)(2.4^a - 5.2^a + 1) < 0 \quad \Rightarrow \quad (2^a - 1)^2 (2.2^{2a} - 5.2^a + 1) < 0$

$\Rightarrow 2.2^{2a} - 5.2^a + 1 < 0 \Rightarrow \frac{1}{2} < 2^a < 1 \Rightarrow \log_2 (1/2) < a < \log_2 1 \Rightarrow -1 < a < 0 \ \therefore \ a \in (-1, 0)$

(b) here coefficient of x^2 is positive

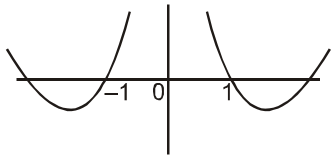
Here $f(x) = 2x^2 - 2(2a + 1)x + a(a + 1)$

$\therefore \quad f(x) = 2a^2 - 2(2a + 1)a + a(a + 1) = -a^2 - a < 0 \Rightarrow -a(a + 1) < 0 \Rightarrow a(a + 1) > 0$

From wavy curve method $a \in (-\infty, -1) \cup (0, \infty)$

Ex.73 Find all values of k for which the inequality, $2x^2 - 4k^2x - k^2 + 1 > 0$ is valid for all real x which do not exceed unity in the absolute value .

Sol. **Case I** when $D < 0$ which gives the result



Case II $f(1) \ge 0$; $D \ge 0$; $-\dfrac{b}{2a} > 1$

or $f(-1) \ge 0$; $D \ge 0$; $-\dfrac{b}{2a} < -1$; No solution in case II. Hence $-\dfrac{1}{\sqrt{2}} < k < \dfrac{1}{\sqrt{2}}$

Ex.74 If $ax^2 - bx + c = 0$ have two distinct roots lying in the interval (0, 1), a, b, c $\in$ N, then prove that $\log_5 abc \ge 2$.

Sol. Let roots are α and β $\therefore$ $\alpha + \beta = b/a$ and $\alpha\beta = c/a$

and $0 < \alpha < 1, 0 < \beta < 1$ $\quad\therefore\quad 0 < 1 - \alpha < 1 \;\&\; < 1 - \beta < 1$

then $\dfrac{\alpha + (1 - \alpha)}{2} \ge \sqrt{\alpha(1 - \alpha)} \Rightarrow \dfrac{1}{4} \ge \alpha(1 - \alpha) > 0$ $\quad\{\because$ A.M. $\ge$ G.M.$\}$

similarly $\dfrac{1}{4} \ge \beta(1 - \beta) > 0 \quad\therefore\quad \dfrac{1}{16} \ge \alpha\beta(1 - \alpha)(1 - \beta) > 0$

But α and β are distinct.

$\therefore \quad 0 < \alpha\beta(1 - \alpha)(1 - \beta) < \dfrac{1}{16} \quad\Rightarrow\quad 0 < \alpha\beta(1 - (\alpha + \beta) + \alpha\beta) < \dfrac{1}{16}$

$\Rightarrow \quad 0 < \dfrac{c}{a}\left(1 - \dfrac{b}{a} + \dfrac{c}{a}\right) < \dfrac{1}{16} \quad\Rightarrow\quad 0 < c(a - b + c) < \dfrac{a^2}{16}$

$\because \quad c(a - b + c)$ = Natural Number ($\because$ a, b, c $\in$ N)

$\therefore \quad$ Minimum value of $c(a - b + c) = 1$

$\therefore \quad \dfrac{a^2}{16} > 1, \quad \therefore a \ge 5 \qquad (\because a \in N)$

and condition for real roots, $\quad b^2 - 4ac \ge 0 \Rightarrow b^2 \ge 4ac$

$\Rightarrow \quad b^2 \ge 20c \qquad (\because a \ge 5)$

$\Rightarrow \quad b \ge 5 \qquad (\because b \in N)$

and minimum value of c = 1

Hence $abc \ge 25 \Rightarrow \log_5(abc) \ge \log_5 5^2 \Rightarrow \log_5(abc) \ge 2$

EXERCISE – I

SINGLE CORRECT (OBJECTIVE QUESTIONS)

1. Number of values 'p' for which the equation $(p^2 - 3p + 2)\, x^2 - (p^2 - 5p + 4)x + p - p^2 = 0$ possess more than two roots, is

(A) 0 (B) 1 (C) 2 (D) None of these

Sol.

2. The roots of the equation

$(b - c)\, x^2 + (c - a)\, x + (a - b) = 0$ are

(A) $\frac{c-a}{b-c}, 1$ (B) $\frac{a-b}{b-c}, 1$ (C) $\frac{b-c}{a-b}, 1$ (D) $\frac{c-a}{a-b}, 1$

Sol.

3. If α, β are the roots of quadratic equation $x^2 + px + q = 0$ and γ, δ are the roots of $x^2 + px - r = 0$, then $(\alpha - \gamma) \,.\, (\alpha - \delta)$ is equal to

(A) $q + r$ (B) $q - r$

(C) $-(q + r)$ (D) $-(p + q + r)$

Sol.

4. Two real numbers α & β are such that $\alpha + \beta = 3$ & $|\alpha - \beta| = 4$, then α & β are the roots of the quadratic equation

(A) $4x^2 - 12x - 7 = 0$ (B) $4x^2 - 12x + 7 = 0$

(C) $4x^2 - 12x + 25 = 0$ (D) None of these

Sol.

5. If a, b, c are integers and $b^2 = 4(ac + 5d^2)$, $d \in N$, then roots of the equation $ax^2 + bx + c = 0$ are

(A) Irrational (B) Rational & defferent

(C) Complex conjugate (D) Rational & equal

Sol.

6. Let a, b and c are real numbers such that $4a + 2b + c = 0$ and $ab > 0$. Then the equation $ax^2 + bx + c = 0$ has

(A) real roots (B) imaginary roots

(C) exactly one root (D) None of these

Sol.

7. Consider the equation $x^2 + 2x - n = 0$, where $n \in N$ and $n \in [5, 100]$. Total number of different values of 'n' so that the given equation has integral roots, is

(A) 4 (B) 6 (C) 8 (D) 3

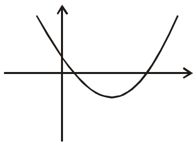
Sol.

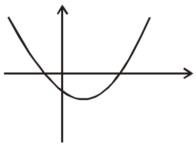
8. If the equation $k\,(6x^2 + 3) + rx + 2x^2 - 1 = 0$ and $6k\,(2x^2 + 1) + px + 4x^2 - 2 = 0$ have both roots common, then the value of $(2r - p)$ is

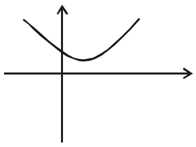
(A) 0 (B) 1/2 (C) 1 (D) None of these

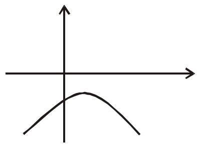
Sol.

9. Which of the following graph represents expression $f(x) = ax^2 + bx + c$ ($a \neq 0$) when $a > 0$, $b < 0$ & $c < 0$?

(A) 

(B) 

(C) 

(D) 

Sol.

10. The expression $y = ax^2 + bx + c$ has always the same sign as of 'a' if

(A) $4ac < b^2$ (B) $4ac > b^2$

(C) $ac < b^2$ (D) $ac > b^2$

Sol.

11. The entire graph of the expression $y = x^2 + kx - x + 9$ is strictly above the x–axis if and only if

(A) $k < 7$ (B) $-5 < k < 7$

(C) $k > -5$ (D) None of these

Sol.

12. If $a, b \in R$, $a \neq 0$ and the quadratic equation $ax^2 - bx + 1 = 0$ has imaginary roots then $a + b + 1$ is

(A) positive (B) negative

(C) zero (D) depends on the sign of b

Sol.

13. If a and b are the non–zero distinct roots of $x^2 + ax + b = 0$, then the least value of $x^2 + ax + b$ is

(A) $\frac{3}{2}$ (B) $\frac{9}{4}$ (C) $-\frac{9}{4}$ (D) 1

Sol.

14. If $y = -2x^2 - 6x + 9$, then

(A) maximum value of y is – 11 and it occurs at x = 2

(B) minimum value of y is – 11 and it occurs at x = 2

(C) maximum value of y is 13.5 and it occurs at x = –1.5

(D) minimum value of y is 13.5 and it occurs atx = –1.5

Sol.

15. Let $f(x) = x^2 + 4x + 1$. Then

(A) $f(x) > 0$ for all x (B) $f(x) > 1$ when $x \geq 0$

(C) $f(x) \geq 1$ when $x \leq -4$ (D) $f(x) = f(-x)$ for all x

Sol.

16. The number of the integer solutions of $x^2 + 9 < (x + 3)^2 < 8x + 25$ is

(A) 1 (B) 2 (C) 3 (D) None of these

Sol.

17. The complete solution set of the inequality $\dfrac{x^4 - 3x^3 + 2x^2}{x^2 - x - 30} \geq 0$ is

(A) $(-\infty, -5) \cup (1, 2) \cup (6, \infty) \cup \{0\}$
(B) $(-\infty, -5) \cup [1, 2] \cup (6, \infty) \cup \{0\}$
(C) $(-\infty, -5) \cup [1, 2] \cup [6, \infty) \cup \{0\}$
(D) None of these

Sol.

18. If the inequality $(m - 2)x^2 + 8x + m + 4 > 0$ is satisfied for all $x \in R$, then least integral m is

(A) 4 (B) 5 (C) 6 (D) None of these

Sol.

19. If α, β are the roots of the quadratic equation $x^2 - 2p(x - 4) - 15 = 0$, then the set of values of p for which one roots is less than 1 & the other root is greater than 2 is

(A) $(7/3, \infty)$ (B) $(-\infty, 7/3)$
(C) $x \in R$ (D) None of these

Sol.

20. If α, β be the roots of $4x^2 - 16x + \lambda = 0$, where $\lambda \in R$ such that $1 < \alpha < 2$ and $2 < \beta < 3$, then the number of integral solutions of λ is

(A) 5 (B) 6 (C) 2 (D) 3

Sol.

21. If two roots of the equation $x^3 - px^2 + qx - r = 0$ are equal in magnitude but opposite in sign, then

(A) $pr = q$ (B) $qr = p$
(C) $pq = r$ (D) None of these

Sol.

22. Let α, β, γ be the roots of $(x - a)(x - b)(x - c) = d$, $d \neq 0$ then the roots of the equation $(x - \alpha)(x - \beta)(x - \gamma) + d = 0$ are

(A) a + 1, b + 1, c + 1 (B) a, b, c
(C) 1 – 1, b – 1, c – 1 (D) $\frac{a}{b}, \frac{b}{c}, \frac{c}{a}$

Sol.

23. If the quadratic equations $3x^2 + ax + 1 = 0$ and $2x^2 + bx + 1 = 0$ have a common root, then the value of the expression $5ab - 2a^2 - 3b^2$ is

(A) 0 (B) 1 (C) –1 (D) None of these

Sol.

24. If both roots of the quadratic equation $(2 - x)(x + 1) = p$ are distinct & positive, then p must lie in the interval

(A) $(2, \infty)$ (B) $(2, 9/4)$ (C) $(-\infty, -2)$ (D) $(-\infty, \infty)$

Sol.

25. The equatiobn $\pi^x = -2x^2 + 6x - 9$ has

(A) no solution (B) one solution

(C) two solutions (D) infinite solutions

Sol.

26. Let a > 0, b > 0 and c > 0. Then both the roots of the equation $ax^2 + bx + c = 0$

(A) are real and negative (B) have negative real parts

(C) are rational numbers (D) have positive real parts

Sol.

27. The set of all solutions of the inequality $(1/2)^{x^2-2x} < 1/4$ contains the set

(A) $(-\infty, 0)$ (B) $(-\infty, 1)$ (C) $(1, \infty)$ (D) $(3, \infty)$

Sol.

28. Consider $y = \frac{2x}{1+x^2}$, where x is real, then the range of expression $y^2 + y - 2$ is

(A) $[-1, 1]$ (B) $[0, 1]$ (C) $[-9/4, 0]$ (D) $[-9/4, 1]$

Sol.

29. The least value of expression $x^2 + 2xy + 2y^2 + 4y + 7$ is

(A) – 1 (B) 1 (C) 3 (D) 7

Sol.

30. The values of k, for which the equation $x^2 + 2(k - 1)x + k + 5 = 0$ possess atleast one positive root, are

(A) $[4, \infty)$ (B) $(-\infty, -1] \cup [4, \infty)$

(C) $[-1, 4]$ (D) $(-\infty, -1]$

Sol.

31. If $\frac{6x^2-5x-3}{x^2-2x+6} \le 4$, then least and the highest values of $4x^2$ are

(A) 0 & 81 (B) 9 & 81
(C) 36 & 81 (D) None of these

Sol.

32. If $\alpha, \beta, \gamma, \delta$ are the roots of the equation $x^4 - Kx^3 + Kx^2 + Lx + M = 0$, where K, L & M are real numbers, then the minimum value of $\alpha^2+\beta^2+\gamma^2+\delta^2$ is

(A) 0 (B) –1 (C) 1 (D) 2

Sol.

33. If the roots of the equation $x^3 + Px^2 + Qx - 19 = 0$ are each one more than the roots of the equation $x^3 - Ax^2 + Bx - C = 0$, where A, B, C, P & Q are constants, then the value of A + B + C is equal to

(A) 18 (B) 19 (C) 20 (D) None of these

Sol.

34. The equations $x^3 + 5x^2 + px + q = 0$ and $x^3 + 7x^2 + px + r = 0$ have tworoots in common. If the third root of each equation is represented by x_1 and x_2 respectivley, then the ordered pair (x_1, x_2) is

(A) (–5, –7) (B) (1, –1)
(C) (–1, 1) (D) (5, 7)

Sol.

35. If coefficients of the equation $ax^2 + bx + c = 0$, $a \neq 0$ are real and roots of the equation are non–real complex and a + c + b < 0, then

(A) 4a + c> 2b (B) 4a + c < 2b
(C) 4a + c = 2b (D) None of these

Sol.

36. If $(\lambda^2 + \lambda - 2)x^2 + (\lambda + 2) x < 1$ for all $x \in R$, then λ belongs to the interval

(A) (–2, 1) (B) $\left[-2, \frac{2}{5}\right)$
(C) $\left(\frac{2}{5}, 1\right)$ (D) None of these

Sol.

37. The set of possible values of λ for which $x^2 - (\lambda^2 - 5\lambda + 5)x + (2\lambda^2 - 3\lambda - 4) = 0$ has roots, whose sum and product are both less than 1, is

(A) $\left(-1, \frac{5}{2}\right)$ (B) (1, 4) (C) $\left[1, \frac{5}{2}\right]$ (D) $\left(1, \frac{5}{2}\right)$

Sol.

38. Let conditions C_1 and C_2 be defined as follows : $C_1 : b^2 - 4ac \geq 0$, C_2 : a, − b, c are of same sign. The roots of $ax^2 + bx + c = 0$ are real and positive, if

(A) bot C_1 and C_2 are satisfied (B) only C_2 is satisfied
(C) only C_1 is satisfied (D) None of these

Sol.

39. If the roots of te equation $x^2 + 2ax + b = 0$ are real and distinct and they differ by at most 2m, then b lies in the interval

(A) $(a^2 - m^2, a^2)$ (B) $[a^2 - m^2, a^2)$
(C) $(a^2, a^2 + m^2)$ (D) None of these

Sol.

40. If the difference between the roots of the equation $x^2 + ax + 1 = 0$ is less than $\sqrt{5}$, then the set of possible values of a is

(A) $(-3, -2] \cup [2, 3)$ (B) $(-3, \infty)$
(C) $(3, \infty)$ (D) $(-\infty, 3)$

Sol.

41. If the roots of the quadratic equation $x^2 + px + q = 0$ are tan 30º and tan 15º respectively, then the value of 2 + q − p is

(A) 3 (B) 0 (C) 1 (D) 2

Sol.

42. All the values of m for which both roots of the equation $x^2 - 2mx + m^2 - 1 = 0$ are greater than −2 but less than 4 lie in the interval

(A) $m > 3$ (B) $-1 < m < 3$
(C) $1 < m < 4$ (D) $-2 < m < 0$

Sol.

43. The value of a for which the sum of the squares of the roots of the equation $x^2 - (a - 2)x - a - 1 = 0$ assume the least value is

(A) 2 (B) 3 (C) 0 (D) 1

Sol.

44. If both the roots of the quadratic equation $x^2 - 2kx + k^2 + k - 5 = 0$ are less than 5, then k lies in the interval
(A) [4, 5] (B) $(-\infty, 4)$ (C) $(6, \infty)$ (D) (5, 6]
Sol.

45. If $(1 - p)$ is root of quadratic equation $x^2 + px + (1 - p) = 0$, then its roots are
(A) 0, 1 (B) −1, 1 (C) 0, −1 (D) −1, 2
Sol.

46. If one root of the equation $x^2 + px + 12 = 0$ is 4, while the equation $x^2 + px + q = 0$ has equal roots then the value of 'q' is
(A) 49/4 (B) 12 (C) 3 (D) 4
Sol.

47. If the sum of the roots of the quadratic equation $ax^2 + bx + c = 0$ is equal to the sum of the squares of their reciprocals, then $\frac{a}{c}, \frac{b}{a}$ and $\frac{c}{b}$ are in
(A) arithmetic progresion
(B) geometric progression.
(C) harmonic progression
(D) arithmetico-geometric progression
Sol.

48. The value of 'a' for which one root of the quadratic equation $(a^2 - 5a + 3)x^2 + (3a - 1)x + 2 = 0$ is twice as large as the other, is
(A) $\frac{2}{3}$ (B) $-\frac{2}{3}$ (C) $\frac{1}{3}$ (D) $-\frac{1}{3}$
Sol.

49. Let a, b, c be real, if $ax^2 + bx + c = 0$ has two real roots α and β, where $\alpha < -2$ and $\beta > 2$, then
(A) $4 + \frac{2b}{a} + \frac{c}{a} = 0$ (B) $4 - \frac{2b}{a} + \frac{c}{a} = 0$
(C) $4 + \frac{2b}{a} - \frac{c}{a} < 0$ (D) $4 - \frac{2b}{a} + \frac{c}{a} < 0$

Sol.

50. The condition for $a^2x^4 + bx^3 + cx^2 + dx + f^2$ may be perfect square is
(A) $2a^2c = a^3f$ (B) $4a^2c - b^2 = 8a^3f$
(C) $4a^3c = 8a^3f$ (D) None of these
Sol.

51. If $a \le 0$ then roots of $x^2 - 2a\,|x - a| - 3a^2 = 0$ is
(A) a (B) $(-1 + \sqrt{6})$ a
(C) $(-\sqrt{6} - 1)$ a (D) None of these
Sol.

52. The values of x and y besides y can satisfy the equation (x, y $\in$ real numbers)
$x^2 - xy + y^2 - 4x - 4y + 16 = 0$
(A) 2, 2 (B) 4, 4 (C) 3, 3 (D) None of these

Sol.

53. For what value of a and b te equation
$x^4 - 4x^3 + ax^2 + bx + 1 = 0$ has four real posuitive roots ?
(A) (–6, –4) (B) (–6, 5)
(C) (–6, 4) (D) (6, –4)
Sol.

54. If α, β are roots of the equation $ax^2 + bx + c = 0$, then the value of $\alpha^3 + \beta^3$ is
(A) $\dfrac{3abc + b^3}{a}$ (B) $\dfrac{a^3 + b^3}{3abc}$
(C) $\dfrac{3abc - b^3}{a^3}$ (D) $\dfrac{-(3abc + b^3)}{a^3}$
Sol.

55. If x is real, then $\frac{x^2 - x + c}{x^2 + x + 2c}$ can take all real values if

(A) $c \in [0, 6]$ (B) $c \in [-6, 0]$

(C) $c \in (-\infty, -6) \cup (0, \infty)$ (D) $c \in (-6, 0)$

Sol.

56. For all $x \in R$, if $mx^2 - 9mx + 5m + 1 > 0$, then m lies in the interval

(A) $-(4/61, 0)$ (B) $[0, 4/61)$

(C) $(4/61, 61/4)$ (D) $(-61/4, 0]$

Sol.

57. If both roots of the quadratic equation $x^2 + x + p = 0$ exceed p, where $p \in R$, then p must lie in the interval

(A) $(-\infty, 1)$ (B) $(-\infty, -2)$

(C) $(-\infty, -2) \cup (0, 1/4)$ (D) $(-2, 1)$

Sol.

58. If a, b, p , q are non-zero real numbers, then two equations, $2a^2x^2 - 2abx + b^2 = 0$ and $p^2x^2 + 2pqx + q^2 = 0$ have

(A) no common root

(B) one common root if $2a^2 + b^2 = p^2 + q^2$

(C) two common roots if $3pq = 2ab$

(D) two common roots if $3qb = 2ap$

Sol.

59. If $a^2 + b^2 + c^2 = 1$, then $ab + bc + ca$ lies in the interval

(A) $\left[\frac{1}{2}, 2\right]$ (B) $[-1, 2]$ (C) $\left[-\frac{1}{2}, 1\right]$ (D) $\left[-1, \frac{1}{2}\right]$

Sol.

60. The value of k for which the equation
$3x^2 + 2x (k^2 + 1) + k^2 - 3k + 2 = 0$ has roots of opposite signs, lies in the interval
(A) $(-\infty, 0)$ (B) $(-\infty, -1)$ (C) $(1, 2)$ (D) $(3/2, 2)$
Sol.

61. If α, β are roots of the equation $ax^2 + bx + c = 0$ then the equation whose roots are $2\alpha + 3\beta$ and $3\alpha + 2\beta$ is
(A) $ab\, x^2 - (a + b)\, cx + (a + b)^2 = 0$
(B) $ac\, x^2 - (a + c)\, bx + (a + c)^2 = 0$
(C) $ac\, x^2 (a + c)\, bx - (a + c)^2 = 0$
(D) None of these
Sol.

62. If two of the roots of the equation
$(a - 1)(x^2 + x + 1)^2 - (a + 1)(x^4 + x^2 + 1) = 0$ are real and distinct, then a lies in the interval
(A) $(-2, 2)$ (B) $(-\infty, -2) \cup (2, \infty)$
(C) $(2, \infty)$ (D) None of these
Sol.

63. If S is the set of all real x such that $\frac{2x-1}{2x^3+3x^2+x}$ is positive, then S contains
(A) $(-\infty, -3/2)$ (B) $(-3/2, 1/4)$
(C) $(-1/4, 1/2)$ (D) $(-1/2, 3)$
Sol.

64. The expression $ax^2 + 2bx + b$ has same sign as that of b for every real x, then the roots of equation $bx^2 + (b - c)\, x + b - c - a = 0$ are
(A) real and equal (B) real and unequal
(C) imaginary (D) None of these
Sol.

65. The number of quadratic equations which are unchanged by squaring their roots is
(A) 2 (B) 4 (C) 6 (D) None of these
Sol.

EXERCISE – II MULTIPLE CORRECT (OBJECTIVE QUESTIONS)

1. If α, β are the roots of $ax^2 + bx + c = 0$, and $\alpha + h$, $\beta + h$ are the roots of $px^2 + qx + r = 0$, then

(A) $\frac{a}{p} = \frac{b}{q} = \frac{c}{r}$ (B) $h = \frac{1}{2}\left(\frac{b}{a} - \frac{q}{p}\right)$

(C) $h = \frac{1}{2}\left(\frac{b}{a} + \frac{q}{p}\right)$ (D) $\frac{b^2 - 4ac}{a^2} = \frac{q^2 - 4pr}{p^2}$

Sol.

2. If a, b are non–zero real numbers and α, β the roots of $x^2 + ax + b = 0$, then

(A) α^2, β^2 are the roots of $x^2 - (2b - a^2)\,x + a^2 = 0$

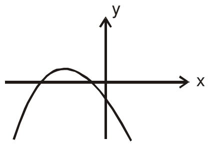
(B) $\frac{1}{\alpha}, \frac{1}{\beta}$ are the roots of $bx^2 + ax + 1 = 0$

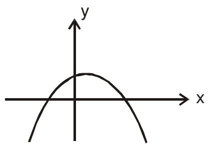
(C) $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$ are the roots of $bx^2 + (2b - a^2)\,x + b = 0$

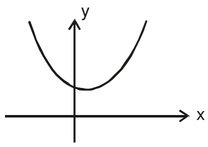
(D) $(\alpha - 1), (\beta - 1)$ are the roots of the equation $x^2 + x\,(a + 2) + 1 + a + b = 0$

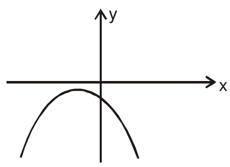
Sol.

3. For which of the following graphs of the quadratic expression $y = ax^2 + b\,x + c$, then product a b c is negative.

(A) 

(B) 

(C) 

(D) 

Sol.

4. $x^2 + x + 1$ is a factor of $a\,x^3 + bx^2 + c\,x + d$, then the real root of above equation is $(a, b, c, d \in R)$

(A) $-d/a$ (B) d/a (C) $(b - a)/a$ (D) $(a - b)/a$

Sol.

5. If the quadratic equations $x^2 + abx + c = 0$ and $x^2 + acx + b = 0$ have a common root, then the equation containing their other roots is/are
(A) $x^2 + a(b + c)\,x + a^2bc = 0$
(B) $x^2 - a(b + c)\,x + a^2bc = 0$
(C) $a(b + c)\,x^2 + (b + c)\,x + abc = 0$
(D) $a(b + c)\,x^2 + (b + c)\,x - abc = 0$
Sol.

6. If the quadratic equations
$ax^2 + bx + c = 0$ $(a, b, c \in R, a \neq 0)$ and $x^2 + 4x + 5 = 0$ have a common root, then a, b, c must satisfy the relations
(A) $a > b > c$ (B) $a < b < c$
(C) $a = k;\ b = 4k;\ c = 5k$ $(k \in R, k \neq 0)$
(D) $b^2 - 4ac$ is negative.
Sol.

7. If α, β are the real and distinct roots of $x^2 + px + q = 0$ and α^4, β^4 are the roots of $x^2 - rx + s = 0$, then the equation $x^2 - 4qx + 2q^2 - r = 0$ has always
(A) two real roots
(B) two negative roots
(C) two positive roots
(D) one positive root and one negative root
Sol.

8. If $(x^2 + x + 1) + (x^2 + 2x + 3) + (x^2 + 3x + 5) + \ldots\ldots + (x^2 + 20\,x + 39) = 4500$, then x is equal to
(A) 10 (B) – 10 (C) 20.5 (D) –20.5
Sol.

9. If roots of equation, $x^3 + bx^2 + cx - 1 = 0$ forms an increasing G.P., then

(A) $b + c = 0$ (B) $b \in (-\infty, -3)$

(C) one of the roots = 1

(D) one root is smaller than 1 & other > 1

Sol.

10. Let $f(x) = \frac{3}{x-2} + \frac{4}{x-3} + \frac{5}{x-4}$, then $f(x) = 0$ has

(A) exactly one real root in (2, 3)

(B) exactly one real root in (3, 4)

(C) at least one real root in (2, 3)

(D) None of these

Sol.

EXERCISE – III SUBJECTIVE QUESTIONS

1. For what value of a, the equation $(a^2 - a - 2)x^2 + (a^2 - 4)x + (a^2 - 3a + 2) = 0$, will have more than two solutions ? Does there exist a real value of x for which the above equation will be an identity in a ?

Sol.

2. If α and β are the roots of the quation $ax^2 + bx + c = 0$, then find the equation whose roots are given by

(i) $\alpha + \frac{1}{\beta}, \beta + \frac{1}{\alpha}$

Sol.

(ii) $\alpha^2 + 2, \beta^2 + 2$

Sol.

3. If $\alpha \neq \beta$ but $\alpha^2 = 5\alpha - 3$, $\beta^2 = 5\beta - 3$, then find the equation whose roots are $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$.

Sol.

4. If $2 + i\sqrt{3}$ is a root of the equation $x^2 + px + q = 0$, where $p, q \in R$, then find the ordered pair (p, q)

Sol.

5. If one root of equation $(l - m)x^2 + lx + 1 = 0$ be double of the other and if l be real, show that $m \leq \frac{9}{8}$.

Sol.

6. Solve the equation $x^4 + 4x^3 + 5x^2 + 2x - 2 = 0$, one root being $-1 + \sqrt{-1}$.

Sol.

7. If the roots of the equation $x^2 - 2cx + ab = 0$ are real and unequal, the prove that the roots of $x^2 - 2(a + b)x + a^2 + b^2 + 2c^2 = 0$ will be imaginary.

Sol.

8. Show that the roots of the quation $(a^2 - bc)x^2 + 2(b^2 - ac)x + c^2 - ab = 0$ are equal if either $b = 0$ or $a^3 + b^3 + c^3 = 3abc$.

Sol.

9. Find the value of the expression $2x^3 + 2x^2 - 7x + 72$ when $x = \frac{3+5\sqrt{-1}}{2}$.

Sol.

10. If $ax^2 + bx + c = 0$ and $bx^2 + cx + a = 0$ have a common root and a, b, c are non–zero real numbers, then find the value of $\frac{a^3+b^3+c^3}{abc}$.

Sol.

11. If one of the roots of the equation $ax^2 + bx + c = 0$ may be reciprocal of one of the roots of $a_1x^2 + b_1x + c_1 = 0$, then prove that $(aa_1 - cc_1)^2 = (bc_1 - ab_1)(b_1c - a_1b)$.

Sol.

12. If x be real, then find the range of the following rational expressions

(i) $y = \frac{x^2+x+1}{x^2+1}$

Sol.

(ii) $y = \frac{x^2-2x+9}{x^2+2x+9}$

Sol.

13. Solve the following inequalities

(i) $x^2 - 7x + 10 > 0$

Sol.

(ii) $x^2 - 4x + 3 < 0$

Sol.

(iii) $\frac{2x}{2x^2+5x+2} > \frac{1}{x+1}$

Sol.

(iv) $\frac{x-2}{x+2} > \frac{2x-3}{4x-1}$

Sol.

14. Solve the following inequalities

(i) $(x - 1)^2 (x + 1)^3 (x - 4) \geq 0$

Sol.

(ii) $\frac{x^4(x+1)^2(x-2)}{(x-3)^3(x+4)} > 0$

Sol.

(iii) $(x^2 - x - 1)(x^2 - x - 7) < -5$

Sol.

(iv) $\frac{(x+2)(x^2-2x+1)}{-4+3x-x^2} \geq 0$

Sol.

15. Find all the values of 'K' for which one root of the equation $x^2 - (K + 1)x + K^2 + K - 8 = 0$, exceeds 2 & the other root is smaller than 2.

Sol.

16. Find the real values of 'a', so that the roots of the equation $(a^2 - a + 2)x^2 + 2(a - 3)x + 9(a^4 - 16) = 0$ are of opposite sign.

Sol.

17. Find all the values of 'a', so that exactly one root of the equation $x^2 - 2ax + a^2 - 1 = 0$, lies between the numbers 2 and 4, and no root of the equation is neither equal to 2 nor equal to 4.

Sol.

18. If α and β be two real roots of the equation $x^3 + px^2 + qx + r = 0$ satisfying the relation $\alpha\beta + 1 = 0$, then prove that $r^2 + pr + q + 1 = 0$.

Sol.

19. If α, β, γ are the roots of the equation $x^3 + px^2 + qx + r = 0$ then find the value of $\left(\alpha - \frac{1}{\beta\gamma}\right)\left(\beta - \frac{1}{\gamma\alpha}\right)\left(\gamma - \frac{1}{\alpha\beta}\right)$.

Sol.

20. If α, β and γ are roots of $2x^3 + x^2 - 7 = 0$, then find the value of $\sum\left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha}\right)$.

Sol.

21. Find all values of the parameter 'a' such that the roots α, β of the equation $2x^2 + 6x + a = 0$ satisfy the inequality $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} < 2$.

Sol.

22. If 'x' is real; find values of 'k' for which, $\left|\frac{x^2 + kx + 1}{x^2 + x + 1}\right| < 2$ is valid $\forall\ x \in R$

Sol.

23. Obtain real solutions of the simultaneous equations, $xy + 3y^2 - x + 4y - 7 = 0$; $2xy + y^2 - 2x - 2y + 1 = 0$.

Sol.

24. If the equations, $x^2 + ax + 12 = 0$, $x^2 + bx + 15 = 0$ & $x^2 + (a + b)x + 36 = 0$ have a common positive root find a & b and the roots of the equations.

Sol.

25. Find all values of a for which atleast one of the roots of the equation $x^2 - (a - 3)x + a = 0$ is greater than 2.

Sol.

26. Find the set of values of 'a' if $(x^2 + x)^2 + a(x^2 + x) + 4 = 0$ has

(i) all four real & distinct roots

Sol.

(ii) Only two roots are real and distinct.

Sol.

(iii) all four roots are imaginary.

Sol.

(iv) four real roots in which only two are equal.

Sol.

27. Prove that roots of $a^2x^2 + (b^2 + a^2 - c^2)x + b^2 = 0$ are not real, if $a + b > c$ and $|a - b| < c$.
(where a, b, c are positive real numbers)

Sol.

28. Find the condition that $\frac{a}{(x-a)}+\frac{b}{(x-b)}$ = m may have roots equal in magnitude but opposite in sign.

Sol.

29. Let $\alpha + i\beta$; $\alpha, \beta \in R$, be a root of the equation $x^3 + qx + r = 0$; q, r $\in$ R . Find a real cubic equation, independent of α and β, whose one root is 2α.

Sol.

30. Find the values of a, for which the quadratic expression $ax^2 + (a - 2) x - 2$ is negative for exactly two integral values of x.

Sol.

31. α, β are the roots of the equation $K(x^2 - x) + x + 5 = 0$. If K_1 & K_2 are the two values of K for which the roots α, β are connected by the relation $(\alpha/\beta) + (\beta/\alpha) = 4/5$. Find the value of $(K_1/K_2) + (K_2/K_1)$.

Sol.

32. If the quadratic equations, $x^2 + bx + c = 0$ and $bx^2 + cx + 1 = 0$ have a common root then prove that either $b + c + 1 = 0$ or $b^2 + c^2 + 1 = bc + b + c$.

Sol.

EXERCISE – IV ADVANCED SUBJECTIVE QUESTIONS

1. If one root of the equation $ax^2 + bx + c = 0$ be the square of the other, prove that $b^3 + a^2c + ac^2 = 3abc$.
Sol.

2. Find the range of values of a, such that
$f(x) = \frac{ax^2 + 2(a+1)x + 9a + 4}{x^2 - 8x + 32}$ is always negative.
Sol.

3. Find a quadratic equation whose sum and product of the roots are the values of the expressions (cosec 10º – $\sqrt{3}$ sec 10º) and (0.5 cosec 10º – 2 sin 70º) respectively. Also express the roots of this quadratic in terms of tangent of an angle lying in $\left(0, \frac{\pi}{2}\right)$.
Sol.

4. Find the least value of $\frac{6x^2 - 22x + 21}{5x^2 - 18x + 17}$ for all real values of x, using the theory of quadratic equations.
Sol.

5. Find the least value of
$(2p^2 + 1) x^2 + 2(4p^2 - 1) x + 4(2p^2 + 1)$ for real values of p and x.
Sol.

6. If α, β are the rotos of the quadratic equation $ax^2 + bx + c = 0$ then which of the following expressions in α, β will denotes the symmetric functions of roots. Give proper reasoning.

(i) $f(\alpha, \beta) = \alpha^2 - \beta$

Sol.

(ii) $f(\alpha,\beta) = \alpha^2\beta + \alpha\beta^2$

Sol.

(iii) $f(\alpha, \beta) = ln\, \dfrac{\alpha}{\beta}$

Sol.

(iv) $f(\alpha, \beta) = \cos(\alpha - \beta)$

Sol.

7. If a & b are positive numbers; prove that the equation $\dfrac{1}{x} + \dfrac{1}{x-a} + \dfrac{1}{x+b} = 0$ has two real roots; one between a/3 & 2a/3 and the other between – 2b/3 & – b/3.

Sol.

8. If the roots of $x^2 - ax + b = 0$ are real & differ by a quantity which is less than c (c > 0), prove that b lies between $(1/4)(a^2 - c^2)$ & $(1/4)a^2$.

Sol.

9. At what values of 'a' do all the zeroes of the function, $f(x) = (a - 2)x^2 + 2ax + a + 3$ lie on the interval (–2, 1) ?

Sol.

10. If the quadratic equations $x^2 + bx + ca = 0$ & $x^2 + cx + ab = 0$; $a \neq 0$, $b \neq c$ have a common root; prove that the equation containing their other root is $x^2 + ax + bc = 0$.

Sol.

11. Find al real numbers x such that,

$$\left(x-\frac{1}{x}\right)^{\frac{1}{2}}+\left(1-\frac{1}{x}\right)^{\frac{1}{2}}=x.$$

Sol.

12. Find the values of 'a' for which
$-3 < [x^2 + ax - 2) / (x^2 + x + 1)] < 2$ is valid for all real x.
Sol.

13. Find the minimum value of $\dfrac{\left(x+\frac{1}{x}\right)^6-\left(x^6+\frac{1}{x^6}\right)-2}{\left(x+\frac{1}{x}\right)^3+x^3+\frac{1}{x^3}}$ for $x > 0$.
Sol.

14. Find the product of the real roots of the equation,
$x^2 + 18x + 30 = 2\sqrt{x^2+18x+45}$
Sol.

15. Let α, β and γ are the roots of the cubic $x^3 - 3x^2 + 1 = 0$. Find a cubic whose roots are $\frac{\alpha}{\alpha-2}, \frac{\beta}{\beta-2}$ and $\frac{\gamma}{\gamma-2}$. Hence or otherwise find the value of $(\alpha - 2)(\beta - 2)(\gamma - 2)$.

Sol.

16. If α, β are roots of the equation, $x^2 - 2x - a^2 + 1 = 0$ and γ, δ are the roots of the equation, $x^2 - 2(a + 1)x + a(a - 1) = 0$ such that $\alpha, \beta \in (\gamma, \delta)$ then find the values of 'a'.

Sol.

17. Two roots of a biquadratic $x^4 - 18x^3 + kx^2 + 200x - 1984 = 0$ have their product equal to (–32). Find the value of k.

Sol.

18. Suppose a cubic polynomial $f(x) = x^3 + px^2 + qx + 72$ is divisible by both $x^2 + ax + b$ and $x^2 + bx + a$ (where a, b, p, q are constants and $a \neq b$). Find the sum of the squares of the roots of the cubic polynomial.

Sol.

19. Find the values of K so that the quadratic equation $x^2 + 2(K - 1)x + K + 5 = 0$ has atleast one positive root.

Sol.

20. Find the values of 'b' for which the equation $2\log_{\frac{1}{25}}(bx + 28) = -\log_5(12 - 4x - x^2)$ has only one solution.

Sol.

21. Find all the values of the parameter 'a' for which both roots of the quadratic equation $x^2 - ax + 2 = 0$ belong to the interval (0, 3).

Sol.

22. Find all the values of the parameters c for which the inequality has at least one solution.

$$1 + \log_2\left(2x^2 + 2x + \frac{7}{2}\right) \geq \log_2(cx^2 + c).$$

Sol.

23. Find the complete set of real values of 'a' for which both roots of the quadratic equation $(a^2 - 6a + 5)x^2 - \sqrt{a^2 + 2a}\ x + (6a - a^2 - 8) = 0$ lie on either side of the origin.

Sol.

24. If $g(x) = x^3 + px^2 + qx + r$ where p, q and r are integers. If g (0) and g(–1) are both odd, then prove that the equation $g(x) = 0$ cannot have three integral roots.

Sol.

25. Find all numbers p for each of which the least value of the quadratic trinomial $4x^2 - 4px + p^2 - 2p + 2$ on the interval $0 \le x \le 2$ is equal to 3.

Sol.

26. Let $P(x) = x^2 + bx + c$, where b and c are integer. If P(x) is a factor of both $x^4 + 6x^2 + 25$ and $3x^4 + 4x^2 + 28x + 5$, find the value of P(1).

Sol.

EXERCISE – V JEE PROBLEMS

1. If α, β are the roots of the equation, $(x-a)(x-b)+c=0$, find the roots of the equation, $(x-\alpha)(x-\beta)=c$. **[REE 2000 (Mains), 3]**

Sol.

2. (a) For the equation, $3x^2+px+3=0$, $p>0$ if one of the roots is square of the other, then p is equal to

(A) 1/3 (B) 1 (C) 3 (D) 2/3

[JEE 2000 (Scr.), 1 + 1 + 1]

Sol.

(b) If α & β ($\alpha < \beta$), are the roots of the equation, $x^2+bx+c=0$, where $c<0<b$, then

(A) $0<\alpha<\beta$ (B) $\alpha<0<\beta<|\alpha|$

(C) $\alpha<\beta<0$ (D) $\alpha<0<|\alpha|<\beta$

Sol.

(c) If $b>a$, then the equation, $(x-a)(x-b)-1=0$, has

(A) both roots in $[a, b]$ (B) both roots in $(-\infty, a)$

(C) both roots in $[b, \infty)$

(D) one root in $(-\infty, a)$ & other in $(b, +\infty)$

Sol.

(d) If α, β are the roots of $ax^2+bx+c=0$, $(a \neq 0)$ and $\alpha+\delta$, $\beta+\delta$, are the roots of $Ax^2+Bx+C=0$, $(A \neq 0)$ for some constant δ, then prove that,

$$\frac{b^2-4ac}{a^2} = \frac{B^2-4AC}{A^2}.$$ **[JEE 2000(Mains), 4]**

Sol.

3. Let a, b, c be real numbers with $a \neq 0$ and let α, β be the roots of the equation $ax^2+bx+c=0$. Express the roots of $a^3x^2+abcx+c^3=0$ in terms of α, β.

[JEE 2001(Mains), 5]

Sol.

4. The set of all real numbers x for which $x^2-|x+2|+x>0$, is **[JEE 2002 (Scr.), 3]**

(A) $(-\infty, -2) \cup (2, \infty)$ (B) $\left(-\infty, -\sqrt{2}\right) \cup \left(\sqrt{2}, \infty\right)$

(C) $(-\infty, -1) \cup (1, \infty)$ (D) $\left(\sqrt{2}, \infty\right)$

Sol.

5. If $x^2 + (a - b)x + (1 - a - b) = 0$ where $a, b \in R$ then find the values of 'a' for which equation has unequal real roots for all values of 'b'.

[JEE 2003 (Mains), 4]

Sol.

6. (a) If one root of the equation $x^2 + px + q = 0$ is the square of the other, then **[JEE 2004 (Scr.)]**

(A) $p^3 + q^2 - q(3p + 1) = 0$
(B) $p^3 + q^2 + q(1 + 3p) = 0$
(C) $p^3 + q^2 + q(3p - 1) = 0$
(D) $p^3 + q^2 + q(1 - 3p) = 0$

Sol.

(b) If $x^2 + 2ax + 10 - 3a > 0$ for all $x \in R$, then

(A) $-5 < a < 2$ (B) $a < -5$
(C) $a > 5$ (D) $2 < a < 5$

Sol.

7. Find the range of values of t for which

$2 \sin t = \dfrac{1 - 2x + 5x^2}{3x^2 - 2x - 1}, t \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$ **[JEE 2005 (Mains), 2]**

Sol.

8. (a) Let a, b, c be the sides of a triangle. No two of them are equal and $\lambda \in R$. If the roots of the equation $x^2 + 2(a + b + c)x + 3\lambda(ab + bc + ca) = 0$ are real, then **[JEE 2006, 3 + 6]**

(A) $\lambda < \frac{4}{3}$ (B) $\lambda > \frac{5}{3}$

(C) $\lambda \in \left(\frac{1}{3}, \frac{5}{3}\right)$ (D) $\lambda \in \left(\frac{4}{3}, \frac{5}{3}\right)$

Sol.

(b) If roots of the equation $x^2 - 10cx - 11d = 0$ are a, b and those of $x^2 - 10ax - 11b = 0$ are c, d, then find the values of a + b + c + d. (a, b, c and d are distinct numbers)

Sol.

9. (a) Let α, β be the roots of the equation $x^2 - px + r = 0$ and $\alpha/2$, 2β be the roots of the equation $x^2 - qx + r = 0$. Then the value of 'r' is **[JEE 2007, 3 + 6]**

(A) $\frac{2}{9}(p - q)(2q - p)$ (B) $\frac{2}{9}(q - p)(2p - q)$

(C) $\frac{2}{9}(q - 2p)(2q - p)$ (D) $\frac{2}{9}(2p - q)(2q - p)$

Sol.

MATCH THE COLUMN

(b) Let $f(x) = \frac{x^2 - 6x + 5}{x^2 - 5x + 6}$

Match the expressions / statements in **Column I** with expressions / statements in **Column II**.

	Column–I		Column–II
(A)	If $-1 < x < 1$, then f(x) satisfies	(P)	$0 < f(x) < 1$
(B)	If $1 < x < 2$, the f(x) satisfies	(Q)	$f(x) < 0$
(C)	If $3 < x < 5$, then f(x) satisfies	(R)	$f(x) > 0$
(D)	If $x > 5$, then f(x) satisfies	(S)	$f(x) < 1$

Sol.

ASSERTION & REASON

10. Let a, b, c, p, q be real numbers. Suppose α, β are the roots of the equation $x^2 + 2px + q = 0$ and α, $1/\beta$ are the roots of the equation $ax^2 + 2bx + c = 0$, where $\beta^2 \notin \{-1, 0, 1\}$ **[JEE 2008, 3]**

STATEMENT–1 : $(p^2 - q)(b^2 - ac) \geq 0$

and

STATEMENT–2 : $b \neq pa$ or $c \neq qa$

(A) Statement-1 is True, Statement-2 is True ; Statement-2 is a correct explanation for Statement-1

(B) Statement-1 is True, Statement-2 is True ; Statement-2 is **NOT** a correct explanation for Statement-1

(C) Statement-1 is True, Statement-2 is False

(D) Statement-1 is False, Statement-2 is True

Sol.

11. The smallest value of k, for which both the roots of the equation, $x^2 - 8kx + 16(k^2 - k + 1) = 0$ are real, distinct and have values at least 4, is **[JEE 2009, 4]**

Sol.

12. Let p and q are real numbers such that $p \neq q$, $p^3 \neq q$ and $p^3 \neq -q$. If α and β are two non zero complex number satisfies $\alpha + \beta = -p$ and $\alpha^3 + \beta^3 = q$ then quadratic equation whose roots are $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$, is **[JEE 2010, 3]**

(A) $(p^3 + q)x^2 - (p^3 + 2q)x + (p^3 + q) = 0$
(B) $(p^3 + q)x^2 - (p^3 - 2q)x + (p^3 + q) = 0$
(C) $(p^3 - q)x^2 - (5p^3 - 2q)x + (p^3 - q) = 0$
(D) $(p^3 - q)x^2 - (5p^3 + 2q)x + (p^3 - q) = 0$

Sol.

13. Let α and β be the roots of $x^2 - 6x - 2 = 0$, with $\alpha > \beta$. If $a_n = \alpha^n - \beta^n$ for $n > 1$, then the value of $\frac{a_{10} - 2a_8}{2a_9}$ is **[JEE 2011, 4]**

(A) 1 (B) 2 (C) 3 (D) 4

Sol.

14. A value of b for which the equations $x^2 + bx - 1 = 0$, $x^2 + x + b = 0$, have one root in common is **[JEE 2011, 4]**

(A) $-\sqrt{2}$ (B) $-i\sqrt{3}$ (C) $i\sqrt{5}$ (D) $\sqrt{2}$

Sol.

Answer Ex–I — SINGLE CORRECT (OBJECTIVE QUESTIONS)

1. B	**2.** B	**3.** C	**4.** A	**5.** A	**6.** A	**7.** C	**8.** A
9. B	**10.** B	**11.** B	**12.** A	**13.** C	**14.** C	**15.** C	**16.** D
17. B	**18.** B	**19.** B	**20.** D	**21.** C	**22.** B	**23.** B	**24.** B
25. A	**26.** B	**27.** D	**28.** C	**29.** C	**30.** D	**31.** A	**32.** B
33. A	**34.** A	**35.** B	**36.** B	**37.** D	**38.** A	**39.** B	**40.** A
41. A	**42.** B	**43.** D	**44.** B	**45.** C	**46.** A	**47.** C	**48.** A
49. D	**50.** B	**51.** B	**52.** B	**53.** D	**54.** C	**55.** D	**56.** B
57. B	**58.** A	**59.** C	**60.** C	**61.** D	**62.** B	**63.** A	**64.** B
65. B							

Answer Ex–II — MULTIPLE CORRECT (OBJECTIVE QUESTIONS)

1. BD	**2.** BCD	**3.** ABCD	**4.** AD	**5.** BD	**6.** CD	**7.** AD
8. AD	**9.** ABCD	**10.** AB				

Answer Ex–III — SUBJECTIVE QUESTIONS

1. a = 2; No real value of x. **2. (i)** $acx^2 + b(a + c)x + (a + c)^2 = 0$ **3.** $3x^2 - 19x + 3 = 0$

(ii) $a^2x^2 + (2ac - 4a^2 - b^2)x + 2b^2 + (c - 2a)^2 = 0$

4. $(-4, 7)$ **6.** $-1 \pm \sqrt{2}$; $-1 \pm \sqrt{-1}$ **9.** 4 **10.** 3 **12. (i)** $\left[\frac{1}{2}, \frac{3}{2}\right]$, **(ii)** $\left[\frac{1}{2}, 2\right]$

13. (i) $x \in (-\infty, 2) \cup (5, \infty)$, **(ii)** $x \in (1, 3)$, **(iii)** $x \in (-2, -1) \cup (-2/3, -1/2)$, **(iv)** $x \in (-\infty, -2) \cup \left(\frac{1}{4}, 1\right) \cup (4, \infty)$

14. (i) $x \in (-\infty, -1] \cup \{1\} \cup [4, \infty)$ **(ii)** $x \in (-4, -1) \cup (-1, 0) \cup (0, 2) \cup (3, \infty)$

(iii) $x \in (-2, -1) \cup (2, 3)$ **(iv)** $x \in (-\infty, -2] \cup \{1\}$

15. $K \in (-2, 3)$ **16.** $a \in (-2, 2)$ **17.** $a \in (1, 5) - \{3\}$ **19.** $-\frac{(r+1)^3}{r^2}$ **20.** -3

21. $(-\infty, 0) \cup (9/2, \infty)$ **22.** $k \in (0, 4)$ **23.** $x \in R$ if $y = 1$, $x = 2$ if $y = -3$

24. a = –7, b = –8; roots (3, 4), (3, 5), (3, 12) **25.** $a \in [9, \infty)$

26. (i) $a \in (-\infty, -4)$ **(ii)** $a \in \left(\frac{65}{4}, \infty\right)$ **(iii)** $a \in \left(-4, \frac{65}{4}\right)$ **(iv)** $a \in \phi$

28. a + b = 0, $m \in (-\infty, -2) \cup (0, \infty)$ or m = –1, ab > 0 **29.** $x^3 + qx - r = 0$ **30.** [1, 2) **31.** 254

Answer Ex–IV ADVANCED SUBJECTIVE QUESTIONS

2. $a \in \left(-\infty, -\frac{1}{2}\right)$ **3.** $x^2 - 4x + 1 = 0$; $\alpha = \tan\left(\frac{\pi}{12}\right)$; $\beta = \tan\left(\frac{5\pi}{12}\right)$ **4.** 1

5. minimum value 3 when x = 1 and p = 0

6. (a) (ii) and (iv) ; **(b)** $x^2 - p(p^4 - 5q^2)\,x + p^2q^2(p^2 - 4q)\,(p^2 - q) = 0$ **9.** $\left(-\infty, -\frac{1}{4}\right) \cup \{2\} \cup (5, 6]$

11. $x = \frac{\sqrt{5}+1}{2}$ **12.** –2 < a < 1 **13.** $y_{min} = 6$ **14.** 20

15. $3y^3 - 9y^2 - 3y + 1 = 0$; $(\alpha - 2)(\beta - 2)(\gamma - 2) = 3$ **16.** $a \in \left(-\frac{1}{4}, 1\right)$ **17.** k = 86

18. 146 **19.** $K \le -1$ **20.** $(-\infty, -14) \cup \{4\} \cup \left[\frac{14}{3}, \infty\right)$ **21.** $2\sqrt{2} \le a < \frac{11}{3}$

22. (0, 8] **23.** $(-\infty, -2] \cup [0, 1) \cup (2, 4) \cup (5, \infty)$ **25.** $a = 1 - \sqrt{2}$ or $5 + \sqrt{10}$ **26.** P(1) = 4

Answer Ex–V JEE PROBLEMS

1. (a, b) **2. (a)** C, **(b)** B, **(c)** D **3.** $\gamma = \alpha^2\beta$ and $\delta = \alpha\beta^2$ or $\gamma = \alpha\beta^2$ and $\delta = \alpha^2\beta$ **4.** B

5. a > 1 **6. (a)** D ; **(b)** A **7.** $\left[-\frac{\pi}{2}, -\frac{\pi}{10}\right] \cup \left[\frac{3\pi}{10}, \frac{\pi}{2}\right]$ **8. (a)** A, **(b)** 1210

9. (a) D, **(b)** (A)–(P),(R),(S) ; (B)–(Q),(S) ; (C)–(Q),(S) ; (D)–(P),(R),(S) **10.** B **11.** 2 **12.** B

13. C **14.** B